

Economics of the Market

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1. Markets, Strategies, and Firms

1.1. Group A

1. Consider a market with demand $d(p) = 100 - 4p$ and supply $s(p) = 40 + 2p$.

- a. Calculate the demanded and supplied quantities for $p = 2$. Does the market clear? Does it have a shortage or a surplus?
- b. Calculate the market clearing price.

Hints

- a. For $p = 2$, we have $d(2) = 92$ and $s(2) = 44$. The market does not clear because $d(2) > s(2)$. It has a shortage equal to $d(2) - s(2) = 48$
 - b. The market clearing price is obtained by setting $d(p) = s(p)$ and solving for p . This gives $p = 10$.
-

2. Consider a market with demand $d(p) = 250/p$ and supply $s(p) = 10p$.

- a. Calculate the market clearing price (assume that the price cannot be negative).
- b. Calculate the price elasticity of demand.
- c. Calculate the price elasticity of supply.

Hints

- a. By $d(p) = s(p)$, we get $p = 5$. The assumption of the exercise excludes the negative root.
- b. The elasticity of demand is given by

$$e_d(p) = d'(p) \frac{p}{d(p)} = -\frac{250}{p^2} \frac{p^2}{250} = -1.$$

- c. Demand elasticity is constant for all prices.
- c. The elasticity of supply is given by

$$e_s(p) = s'(p) \frac{p}{s(p)} = 10 \frac{p}{10p} = 1.$$

Supply elasticity is also constant for all prices.

3. Consider a market with demand $d(p) = 22 - 3p$ and supply $s(p) = 10 + p$.

- Calculate the market clearing price.
- Calculate the price elasticity of demand at the market clearing price.
- Calculate the price elasticity of supply at the market clearing price.

Hints

- We calculate $p = 3$.
- The elasticity of demand at $p = 3$ is

$$e_d(3) = -3 \frac{p}{22 - 3p} \Big|_{p=3} = -\frac{9}{13}.$$

- The elasticity of supply at $p = 3$ is

$$e_s(3) = \frac{p}{10 + p} \Big|_{p=3} = \frac{3}{13}.$$

1.2. Group B

1. Consider a market with demand $d(p) = \alpha - \beta p$ and supply $s(p) = \gamma + \delta p$, where all Greek letters are positive parameters.

- Calculate the market clearing price.
- Calculate the price elasticity of demand at the market clearing price.
- Calculate the price elasticity of supply at the market clearing price.

Hints

- The market clearing price is given by $d(p) = s(p)$, or equivalently

$$\alpha - \beta p = \gamma + \delta p.$$

We can solve for prices to get

$$p_m = \frac{\alpha - \gamma}{\delta + \beta}.$$

- The elasticity of demand at p_m is

$$e_d(p_m) = -\beta \frac{1}{\frac{\alpha}{p_m} - \beta} = -\beta \frac{\alpha - \gamma}{\alpha\delta + \beta\gamma}.$$

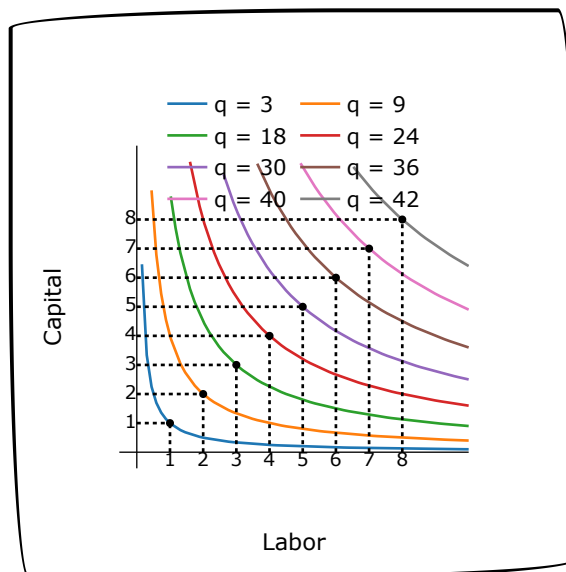
- The elasticity of supply at p_m is

$$e_s(p_m) = \delta \frac{1}{\frac{\gamma}{p_m} + \delta} = \delta \frac{\alpha - \gamma}{\alpha\delta + \beta\gamma}.$$

2. Technology

2.1. Group A

- The following graph plots the isoquants of a production function for various levels of output. Determine the regions where the production function exhibits constant, increasing, and decreasing returns to scale.



Hints

Initial q	Final q	Input scale	Final q / Initial q	Returns to scale
3	9	2	3	Increasing
9	18	1.5	2	Increasing
18	24	1.3333333	1.3333333	Constant
24	30	1.25	1.25	Constant
30	36	1.2	1.2	Constant
36	40	1.1666667	1.1111111	Decreasing
40	42	1.1428571	1.05	Decreasing

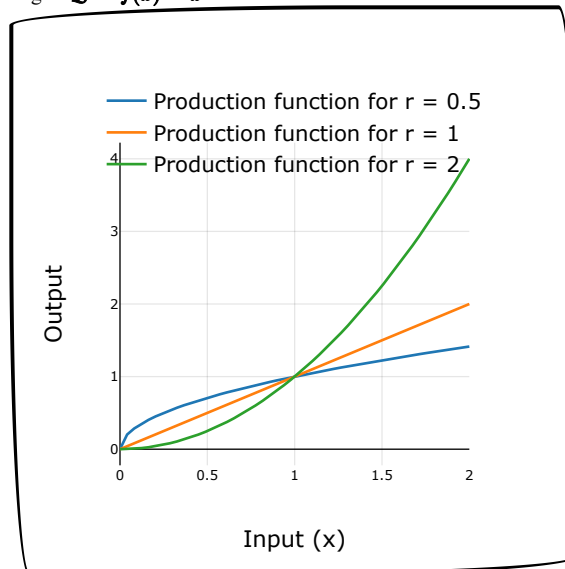
2. Consider the family of production sets

$$\mathcal{Q}^r = \{(x, q) \in \mathbb{R}_{\geq 0}^2 : 0 \leq q \leq x^r\} \quad (0 < r < \infty).$$

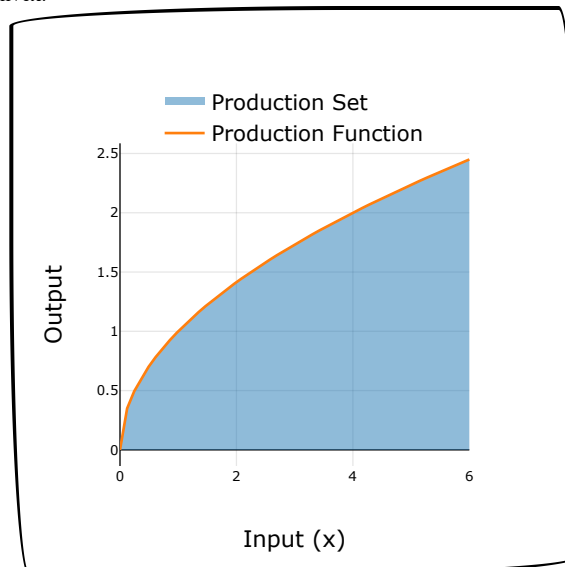
- Plot in a single graph the production functions for $0 < r_1 < 1 = r_2 < r_3$.
- Sketch the production set for a fixed $r < 1$.
- Sketch the production set for a fixed $r > 1$.
- For which values of r the marginal product of the production function corresponding to $\mathcal{Q}^r$ is diminishing / constant / increasing?

Hints

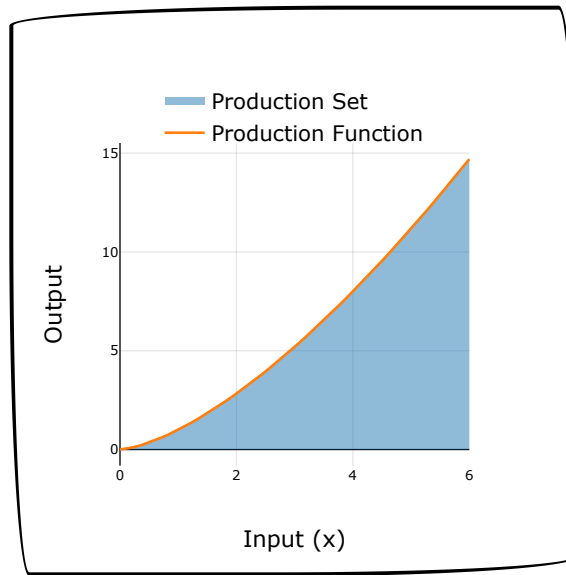
- The production function corresponding to $\mathcal{Q}^r$ is $f(x) = x^r$.



- For $r < 1$, the production set is convex.



- For $r > 1$, the production set is not convex.



- d. Given any $r > 0$, the marginal product of the production function is $f'(x) = rx^{r-1}$.
- If $r < 1$, then $f''(x) = r(r-1)x^{r-2} < 0$, which implies that the marginal product of f is decreasing.
 - If $r = 1$, then $f''(x) = r(r-1)x^{r-2} = 0$, which implies that the marginal product of f is constant.
 - If $r > 1$, then $f''(x) = r(r-1)x^{r-2} > 0$, which implies that the marginal product of f is increasing.

2.2. Group B

1. Consider the production function $f(x_1, x_2) = x_1^\alpha x_2^\beta$ for parameters $\alpha, \beta > 0$. For which values of α and β does f exhibit decreasing / constant / increasing returns to scale?

Hints

For every $t > 1$, we have

$$\begin{aligned} f(tx_1, tx_2) &= (tx_1)^\alpha (tx_2)^\beta \\ &= t^{\alpha+\beta} x_1^\alpha x_2^\beta \\ &= t^{\alpha+\beta} f(x_1, x_2). \end{aligned}$$

- If $\alpha + \beta > 1$, then $t^{\alpha+\beta} > t$ and $f(tx_1, tx_2) > tf(x_1, x_2)$, which means that f exhibits increasing returns to scale.
- If $\alpha + \beta < 1$, then $t^{\alpha+\beta} < t$ and $f(tx_1, tx_2) < tf(x_1, x_2)$, which means that f exhibits decreasing returns to scale.
- If $\alpha + \beta = 1$, then $t^{\alpha+\beta} = t$ and $f(tx_1, tx_2) = tf(x_1, x_2)$, which means that f exhibits constant returns to scale.

2.3. Group C

1. Show that the elasticity of the constant elasticity production function is constant and equal to $(1 - \rho)^{-1}$.

Hints

The marginal product of the constant elasticity production function with respect to the first input factor is

$$\begin{aligned} \frac{\partial f(x_1, x_2)}{\partial x_1} &= A \frac{r}{\rho} (\alpha x_1^\rho + (1 - \alpha)x_2^\rho)^{\frac{r}{\rho}-1} \alpha \rho x_1^{\rho-1} \\ &= \frac{f(x_1, x_2)}{\alpha x_1^\rho + (1 - \alpha)x_2^\rho} \alpha r x_1^{\rho-1}. \end{aligned}$$

Analogously, the marginal product with respect to the second input factor is

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{f(x_1, x_2)}{\alpha x_1^\rho + (1 - \alpha)x_2^\rho} (1 - \alpha) r x_2^{\rho-1}.$$

One can then calculate the marginal rate of technical substitution by

$$\begin{aligned} \text{MRTS}(x_1, x_2) &= \frac{\partial f(x_1, x_2) / \partial x_1}{\partial f(x_1, x_2) / \partial x_2} \\ &= \frac{\alpha}{1 - \alpha} \left(\frac{x_1}{x_2} \right)^{\rho-1}, \end{aligned}$$

and, thus,

$$\ln \text{MRTS}(x_1, x_2) = \ln \frac{\alpha}{1 - \alpha} + (\rho - 1) \ln \frac{x_1}{x_2}.$$

Then, we have

$$\frac{d \ln \text{MRTS}(x_1, x_2)}{d \ln(x_2/x_1)} = 1 - \rho,$$

which implies that

$$e(x_2, x_1) = \frac{1}{1 - \rho}.$$

2. Consider the constant elasticity of substitution function

$$f(x_1, x_2) = A(\alpha x_1^\rho + (1 - \alpha)x_2^\rho)^{\frac{1}{\rho}}.$$

- Show that for $\rho \rightarrow -\infty$ f converges to the fixed proportion production function $f(x_1, x_2) \rightarrow A \min\{x_1, x_2\}^r$.
- Show that for $\rho = 1$ f reduces to the perfect substitutes production function $f(x_1, x_2) = A(\alpha x_1 + (1 - \alpha)x_2)^r$.
- Show that for $\rho \rightarrow 0$ f converges to the Cobb-Douglas production function $f(x_1, x_2) \rightarrow Ax_1^{\alpha r} x_2^{(1-\alpha)r}$.

Hints

- Suppose that $x_1 = \min\{x_1, x_2\}$. For every $\rho < 0$, we can rewrite the constant elasticity of substitution function as

$$f(x_1, x_2) = Ax_1^r \left(\alpha + (1 - \alpha) \left(\frac{x_2}{x_1} \right)^\rho \right)^{\frac{1}{\rho}}.$$

Since $x_2/x_1 > 1$, we have $0 \leq (x_2/x_1)^\rho \leq 1$ for all $\rho < 0$. Hence

$$Ax_1^r \alpha^{\frac{1}{\rho}} \leq f(x_1, x_2) \leq Ax_1^r,$$

which implies

$$Ax_1^r = \lim_{\rho \rightarrow -\infty} Ax_1^r \alpha^{\frac{1}{\rho}} \leq \lim_{\rho \rightarrow -\infty} f(x_1, x_2) \leq \lim_{\rho \rightarrow -\infty} Ax_1^r = Ax_1^r.$$

When $x_2 = \min\{x_1, x_2\}$, a similar argument shows that

$$\lim_{\rho \rightarrow -\infty} f(x_1, x_2) = Ax_2^r.$$

We can combine the two cases by writing

$$\lim_{\rho \rightarrow -\infty} f(x_1, x_2) = A \min\{x_1, x_2\}^r.$$

- The result is obtained by substituting $\rho = 1$.
- We can rewrite

$$f(x_1, x_2) = A \exp \left(r \frac{\ln(\alpha x_1^\rho + (1 - \alpha)x_2^\rho)}{\rho} \right).$$

By L'Hospital's

$$\begin{aligned} \lim_{\rho \rightarrow 0} \frac{\ln(\alpha x_1^\rho + (1 - \alpha)x_2^\rho)}{\rho} &= \lim_{\rho \rightarrow 0} \frac{\alpha x_1^\rho \ln x_1 + (1 - \alpha)x_2^\rho \ln x_2}{\alpha x_1^\rho + (1 - \alpha)x_2^\rho} \\ &= \alpha \ln x_1 + (1 - \alpha) \ln x_2 \\ &= \ln x_1^\alpha + \ln x_2^{(1-\alpha)}. \end{aligned}$$

By the continuous mapping theorem, one can then conclude

$$\begin{aligned} \lim_{\rho \rightarrow 0} f(x_1, x_2) &= A \exp \left(r \ln x_1^\alpha + r \ln x_2^{(1-\alpha)} \right) \\ &= Ax_1^{\alpha r} x_2^{(1-\alpha)r}. \end{aligned}$$

3. Profit Maximization

3.1. Group A

- Suppose that a firm produces output q using only labor L according to the production function $q = \sqrt{L}$. The price of output is $p = 2$, and the price of labor is $w = 1$. Find the profit maximizing labor and the corresponding output. What is the optimal profit?

Hints

Maximize $\pi(L) = 2\sqrt{L} - L$ to find $L = 1$ and $q = 1$. Then, the optimal profit is $\pi = 1$.

- Suppose that a firm produces output q using labor L and capital K according to the production function $q = K^{1/4}L^{1/4}$. The price of output is $p = 2$, the wage is $w = 1$, and the interest rate is $r = 1$.

- What happens to output when capital and labor are doubled?
- Find the profit maximizing labor, capital, and output.
- Find the maximum profits.
- Is there a unique maximizing output if the production is given by $q = K^{1/2}L^{1/2}$? Why, or why not?

Hints

- The production function exhibits decreasing returns to scale.
- Maximize $\pi(K, L) = 2K^{1/4}L^{1/4} - K - L$ to find $L = K$. Substituting into the production function, we find that the maximum satisfies $\pi(L) = 2L^{1/2} - 2L$. Not all of these points are maximizing allocations. Maximizing $\pi(L) = 2L^{1/2} - 2L$ gives $L = 1/4$. Thus, $K = 1/4$ and $q = 1/2$.
- Substituting $L = K = 1/4$ into the profit function gives $\pi = 1/2$.
- If we attempt to follow the same process, we find $\pi(L) = 2L - 2L = 0$. Thus, profits are zero irrespective of the chosen allocation. Every choice of output produced by $K = L$ constitutes a maximizing allocation. This is because the production function exhibits constant returns to scales.

3.2. Group B

- Consider a firm with a two dimensional production function f .

- What happens to the firm's profits if f exhibits increasing returns to scale and the firm doubles its scale of operation while all prices remain fixed?
- What happens to the firm's profits if f exhibits decreasing returns to scale and the firm halves its scale of operation while all prices remain fixed?

Hints

- The profits of the firm more than double because the output more than doubles, while the cost doubles.

$$\begin{aligned}\pi(2x_1, 2x_2) &= pf(2x_1, 2x_2) - w_1 2x_1 - w_2 2x_2 \\ &> 2(pf(x_1, x_2) - w_1 x_1 - w_2 x_2) \\ &= 2\pi(x_1, x_2).\end{aligned}$$

- The profits of the firm less than halve because the output less than halves, while the cost halves.

$$\begin{aligned}\pi(x_1, x_2) &= \pi\left(2\frac{1}{2}x_1, 2\frac{1}{2}x_2\right) \\ &= pf\left(2\frac{1}{2}x_1, 2\frac{1}{2}x_2\right) - w_1 2\frac{1}{2}x_1 - w_2 2\frac{1}{2}x_2 \\ &< 2\left(pf\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right) - w_1 \frac{1}{2}x_1 - w_2 \frac{1}{2}x_2\right) \\ &= 2\pi\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right).\end{aligned}$$

Hence,

$$\pi\left(\frac{1}{2}x_1, \frac{1}{2}x_2\right) > \frac{1}{2}\pi(x_1, x_2).$$

- Consider a firm with a single input factor production technology f having diminishing marginal product. The output and input prices, p and w , are fixed. The firm produces at a positive output level for which $pf'(x) > w$ holds and makes positive profits. Is this a profit maximizing firm? If not, and the firm wishes to increase its profits, should it increase or decrease the amount of the input factor?

Hints

Since the firm makes positive profits, it is better off than not producing at all, in which case its profits are non-positive. However, the firm is not producing the profit maximizing output because the variational condition of interior solutions, namely $pf'(x) = w$, is not satisfied. Since p and w are fixed, $f'(x)$ needs to decrease for the variational condition to hold. This implies that the input factor level should increase because the marginal product is decreasing.

- Consider a firm with a production function $f(x_1, x_2) = x_1^{1/2}x_2^{1/4}$. The output price is p , and the input prices are w_1 and w_2 . All prices are fixed.

- Show that, when maximizing profits, the marginal product of each factor is equal to the corresponding real factor price.
- Use your answer in the previous part to calculate the input factor demands.
- Suppose that $p = 4$, $w_1 = 2$, and $w_2 = 1$. What are the optimal demanded quantities of the input factors? What is the optimal supplied quantity? How much profit does the firm make?

Hints

- The maximization problem of the firm is

$$\max_{x_1, x_2} \{px_1^{1/2}x_2^{1/4} - w_1x_1 - w_2x_2\},$$

with first order conditions

$$\underbrace{\frac{w_1}{p}}_{\text{Real factor price}} = \frac{1}{2} x_1^{-1/2} x_2^{1/4}$$

$$\underbrace{\frac{w_2}{p}}_{\text{Real factor price}} = \frac{1}{4} x_1^{1/2} x_2^{-3/4} \underbrace{\quad}_{\text{Marginal product}}$$

b. Multiplying the two necessary conditions results in

$$\frac{w_1 w_2}{p^2} = \frac{1}{8} x_1^{1/2-1/2} x_2^{1/4-3/4},$$

and solving for x_2 gives

$$x_2 = \frac{p^4}{64 w_1^2 w_2^2}.$$

To obtain the factor demand for x_1 , we substitute the demand for x_2 in the first variational condition to get

$$\frac{w_1}{p} = \frac{1}{2} x_1^{-1/2} \frac{p}{2\sqrt{2} w_1^{1/2} w_2^{1/2}}.$$

Then solving for x_1 gives

$$x_1 = \frac{p^4}{32 w_1^3 w_2}.$$

c. Lastly, substituting the prices into the demands gives $x_1 = x_2 = 1$. Substituting the optimal demanded quantities into the production function gives $q = f(1, 1) = 1$. Lastly, the firm's profit is calculated as $\pi = 4 - 2 - 1 = 1$.

4. Suppose that at the beginning of a given month, we observe that a firm produces an output level $q = 20$ at a price level $p = 2$. After a month, we observe that the same firm produces $q = 15$ at a price of $p = 4$. No other changes are observed in the prices of the market. Does the firm maximize its profit?

Hints

Since all input prices remain constant, we have $\Delta w_i = 0$ for all input factors between the two dates. This implies that the revealed profit maximization inequality reduces to $\Delta p \Delta q \geq 0$. However, we observe $\Delta p = 2$ and $\Delta q = -5$. Thus, by the revealed profit maximization inequality, the firm did not maximize its profit on one of these two (or both) dates.

3.3. Group C

1. Consider a firm with a Cobb-Douglas production function $f(x_1, x_2) = A x_1^\alpha x_2^{(1-\alpha)r}$ for $0 < \alpha < 1$ and $r \neq 1$. Output and input prices, namely p , w_1 , and w_2 , are fixed.
- Write the profit maximization problem of the firm.
 - Calculate the first order conditions of the profit maximization problem.
 - Calculate the supply function.
 - Calculate the input factor demands.
 - Calculate the profit of the firm.
 - What happens to supply, input factor demands, and profit for $r = 1$?

Hints

- a. The maximization problem of the firm is

$$\max_{x_1, x_2} \{ p A x_1^\alpha x_2^{(1-\alpha)r} - w_1 x_1 - w_2 x_2 \}$$

- b. The necessary conditions for interior solutions are

$$w_1 = p A \alpha x_1^{\alpha-1} x_2^{(1-\alpha)r} = p \alpha r \frac{f(x_1, x_2)}{x_1}$$

$$w_2 = p(1-\alpha)r x_1^\alpha x_2^{(1-\alpha)r-1} = p(1-\alpha)r \frac{f(x_1, x_2)}{x_2}$$

- c. To shorten the notation, let $q = f(x_1, x_2)$. With this notation, we can rewrite the first order conditions as

$$x_1 = p \alpha r \frac{q}{w_1}$$

$$x_2 = p(1-\alpha)r \frac{q}{w_2}.$$

Substituting in the production function gives

$$q = f(x_1, x_2)$$

$$= A \left(p \alpha r \frac{q}{w_1} \right)^\alpha \left(p(1-\alpha)r \frac{q}{w_2} \right)^{(1-\alpha)r}$$

$$= A (p r q)^r \left(\frac{\alpha}{w_1} \right)^\alpha \left(\frac{1-\alpha}{w_2} \right)^{(1-\alpha)r},$$

which implies

$$q(w_1, w_2) = A^{\frac{1}{1-r}} (pr)^{\frac{r}{1-r}} \left(\frac{\alpha}{w_1} \right)^{\frac{\sigma r}{1-r}} \left(\frac{1-\alpha}{w_2} \right)^{\frac{(1-\sigma)r}{1-r}}.$$

d. We can obtain the input factor demands by replacing the supply function in the first order conditions. This gives

$$\begin{aligned} x_1 &= p\alpha r \frac{A^{\frac{1}{1-r}} (pr)^{\frac{r}{1-r}} \left(\frac{\alpha}{w_1} \right)^{\frac{\sigma r}{1-r}} \left(\frac{1-\alpha}{w_2} \right)^{\frac{(1-\sigma)r}{1-r}}}{w_1} \\ &= (Apr)^{\frac{1}{1-r}} \left(\frac{\alpha}{w_1} \right)^{\frac{1-(1-\sigma)r}{1-r}} \left(\frac{1-\alpha}{w_2} \right)^{\frac{(1-\sigma)r}{1-r}}, \end{aligned}$$

and analogously

$$x_2 = (Apr)^{\frac{1}{1-r}} \left(\frac{\alpha}{w_1} \right)^{\frac{\sigma r}{1-r}} \left(\frac{1-\alpha}{w_2} \right)^{\frac{1-\sigma r}{1-r}}.$$

e. The easiest way to calculate the profit of the firm is to use the first order conditions of part 2 and the supply function. We then have

$$\begin{aligned} \pi(p, w_1, w_2) &= pq - w_1 p\alpha r \frac{q}{w_1} - w_2 p(1-\alpha)r \frac{q}{w_2} \\ &= pq(1-\alpha r - (1-\alpha)r) \\ &= p(1-r)q \\ &= (pA)^{\frac{1}{1-r}} (1-r)r^{\frac{r}{1-r}} \left(\frac{\alpha}{w_1} \right)^{\frac{\sigma r}{1-r}} \left(\frac{1-\alpha}{w_2} \right)^{\frac{(1-\sigma)r}{1-r}}. \end{aligned}$$

f. If $r = 1$, the production technology has constant returns to scale. Then, the firm has zero profit, and it is indifferent about its level of supply.

4. Cost Minimization

4.1. Group A

1. Consider a firm with a production function $f(x_1, x_2) = 4x_1^{1/2}x_2^{1/2}$. The input prices are $w_1 = 40$ and $w_2 = 10$.

- What is the slope of the isocost lines with respect to the first input factor?
- What is the cost minimizing marginal rate of technical substitution between x_1 and x_2 ?
- Find the cost minimizing conditional factor demanded quantities and the minimum production cost when the firm produces an output level q .
- Calculate the cost minimizing conditional factor demanded quantities and the minimum production cost when $q = 40$.

Hints

a. For a fixed level of cost $\hat{c}$, we have $\hat{c} = 40x_1 + 10x_2$. Solving for x_2 gives

$$x_2 = \frac{1}{10}\hat{c} - 4x_1,$$

so the slope of the isocost with respect to x_1 is equal to -4 .

b. From the cost minimizing condition, one can calculate

$$\text{MRTS}(x_1, x_2) = \frac{w_1}{w_2} = 4.$$

The firm substitutes a unit of x_1 with 4 units of x_2 at the minimizing cost level of production.

c. Using the cost minimizing condition, we obtain

$$4 = \text{MRTS}(x_1, x_2) = \frac{f_{x_1}(x_1, x_2)}{f_{x_2}(x_1, x_2)} = \frac{2x_1^{-1/2}x_2^{1/2}}{2x_1^{1/2}x_2^{-1/2}} = \frac{x_2}{x_1}.$$

Solving for x_2 and substituting into the production function gives

$$q = 4x_1^{1/2}x_2^{1/2} = 4x_1^{1/2}2x_1^{1/2} = 8x_1.$$

The last equation implies that $x_1 = q/8$ and $x_2 = q/2$. Hence, the cost function is

$$c(q) = w_1x_1 + w_2x_2 = 40\frac{q}{8} + 10\frac{q}{2} = 10q.$$

d. Substituting the given output level results in $x_1 = 5$ and $x_2 = 20$. Moreover, the minimum cost is 400.

2. Consider a firm with a production technology f having two factors that exhibit diminishing marginal rates of technical substitution. The input factor prices, w_1 and w_2 , are fixed. The firm produces an output level q using input factors quantities such that $\text{MP}(x_1; x_2)/w_1 > \text{MP}(x_2; x_1)/w_2$ holds. Is this a cost minimizing firm? If not, how can the firm reduce its cost?

Hints

The firm does not minimize its cost because the variational condition $\text{MRTS}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{w}_1/\mathbf{w}_2$ is not satisfied. Since $\mathbf{w}_1$ and $\mathbf{w}_2$ are fixed, $\text{MRTS}(\mathbf{x}_1, \mathbf{x}_2)$ needs to decrease for the variational condition to hold. Since the marginal rate of technical substitution is decreasing, the firm can reduce its cost by increasing the input factor level of $\mathbf{x}_1$ and decreasing that of $\mathbf{x}_2$.

3. Suppose that at the beginning of a given month, we observe that a firm produces an output level q using input factors $\mathbf{x}_1 = 1$ and $\mathbf{x}_2 = 3$ with corresponding prices $\mathbf{w}_1 = 3$ and $\mathbf{w}_2 = 1$. After a month, we observe that the same firm produces the same level of output using $\mathbf{x}_1 = 5$ and $\mathbf{x}_2 = 1$ under prices $\mathbf{w}_1 = 5$ and $\mathbf{w}_2 = 3$. Does the firm minimize its cost?

Hints

We calculate $\Delta \mathbf{w}_1 = 2$, $\Delta \mathbf{w}_2 = 2$, $\Delta \mathbf{x}_1 = 4$, and $\Delta \mathbf{x}_2 = -2$. Therefore

$$\Delta \mathbf{w}_1 \Delta \mathbf{x}_1 + \Delta \mathbf{w}_2 \Delta \mathbf{x}_2 = 2 \cdot 4 + 2 \cdot (-2) = 4 > 0.$$

Since the revealed cost minimization inequality is not satisfied, the firm does not minimize its cost at one of these two (or both) dates.

4. Consider a firm with a Cobb-Douglas production function $f(\mathbf{x}_1, \mathbf{x}_2) = 2\mathbf{x}_1^{\frac{1}{6}}\mathbf{x}_2^{\frac{2}{3}}$. Input prices, namely $\mathbf{w}_1$, and $\mathbf{w}_2$, are fixed.

- What returns to scale does the production function have?
- Write the Lagrangian of the cost minimization problem of the firm.
- Calculate the first order conditions of the cost minimization problem.
- Calculate the conditional input factor demands.
- Calculate the cost function of the firm.

Hints

- The production function exhibits decreasing returns to scale.
- The Lagrangian of the problem is obtained by subtracting from the expenditure of the firm the production constraint multiplied by the Lagrange multiplier λ , i.e.,

$$\mathcal{L} = \mathbf{w}_1 \mathbf{x}_1 + \mathbf{w}_2 \mathbf{x}_2 - \lambda \left(2\mathbf{x}_1^{\frac{1}{6}}\mathbf{x}_2^{\frac{2}{3}} - q \right).$$

- The first order conditions are obtained by differentiating the Lagrangian with respect to the two input factors ($\mathbf{x}_1$ and $\mathbf{x}_2$). This gives

$$\begin{aligned} \mathbf{w}_1 &= \frac{2\lambda \mathbf{x}_1^{-\frac{5}{6}} \mathbf{x}_2^{\frac{2}{3}}}{6}, \\ \mathbf{w}_2 &= \frac{2\lambda \mathbf{x}_1^{\frac{1}{6}} \mathbf{x}_2^{-\frac{1}{3}}}{3}. \end{aligned}$$

- From the two first order conditions, we obtain

$$\frac{\mathbf{w}_1}{\mathbf{w}_2} = \frac{1}{2} \frac{\mathbf{x}_2}{\mathbf{x}_1}.$$

Solving for $\mathbf{x}_1$ and substituting to the production function gives the conditional demand of $\mathbf{x}_2$, namely

$$\mathbf{x}_2 = 2^{-\frac{5}{3}} \left(\frac{\mathbf{w}_1}{\mathbf{w}_2} \right)^{\frac{1}{3}} q^2.$$

Instead, the conditional demand of $\mathbf{x}_1$ is obtained by solving for $\mathbf{x}_2$ and substituting to the production function, i.e.,

$$\mathbf{x}_1 = 2^{-\frac{8}{3}} \left(\frac{\mathbf{w}_2}{\mathbf{w}_1} \right)^{\frac{2}{3}} q^2.$$

- The cost function is calculated by replacing the conditional factor demands in the expenditure of the firm, namely

$$\begin{aligned} c(q, \mathbf{w}_1, \mathbf{w}_2) &= \mathbf{w}_1 2^{-\frac{5}{3}} \left(\frac{\mathbf{w}_2}{\mathbf{w}_1} \right)^{\frac{1}{3}} q^2 + \mathbf{w}_2 2^{-\frac{8}{3}} \left(\frac{\mathbf{w}_1}{\mathbf{w}_2} \right)^{\frac{2}{3}} q^2 \\ &= \frac{3}{2^{8/3}} \mathbf{w}_1^{\frac{1}{3}} \mathbf{w}_2^{\frac{2}{3}} q^2. \end{aligned}$$

4.2. Group B

- Prove that if a firm maximizes its profits, then it minimizes its costs.

Hints

Suppose that a firm maximizes its profits under prices p , $\mathbf{w}_1$, $\mathbf{w}_2$ when producing $\hat{q}$ and demanding $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$. Towards contradiction, suppose that $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ do not minimize the firm's cost for producing $\hat{q}$ under the given prices. Then, there exist $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$, different than $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$, that produce $\hat{q}$ with less cost. Then

$$\begin{aligned} \mathbf{w}_1 \hat{\mathbf{x}}_1 + \mathbf{w}_2 \hat{\mathbf{x}}_2 &> \mathbf{w}_1 \tilde{\mathbf{x}}_1 + \mathbf{w}_2 \tilde{\mathbf{x}}_2 \implies \\ \hat{\pi} = p\hat{q} - \mathbf{w}_1 \hat{\mathbf{x}}_1 - \mathbf{w}_2 \hat{\mathbf{x}}_2 &< p\hat{q} - \mathbf{w}_1 \tilde{\mathbf{x}}_1 - \mathbf{w}_2 \tilde{\mathbf{x}}_2, \end{aligned}$$

which implies that $\hat{q}$, $\hat{x}_1$, and $\hat{x}_2$ are not profit maximizing.

2. What are the differences between the conditional factor demand and the factor demand? How are they related?

Hints

The conditional factor demand is obtained by the cost minimization problem, while the factor demand is obtained by the profit maximization problem. The conditional factor demand depends on the output quantity, say q , and factor prices, say w_i . Instead, the factor demand depends on the output price, say p , and factor prices.

Let x_i^c and x_i correspondingly denote the conditional factor and factor demands. By exercise 1, if a firm is maximizing its profits, then it minimizes its costs. Therefore, if $q(p, w_1, w_2)$ is the supply function obtained by the profit maximization problem, then the two demands are interrelated by

$$x_i^c(q(p, w_1, w_2), w_1, w_2) = x_i(p, w_1, w_2).$$

3. Consider a firm with a Cobb-Douglas production function $f(x_1, x_2) = Ax_1^{\alpha}x_2^{(1-\alpha)r}$ for $0 < \alpha < 1$ and $r \neq 1$. Input prices, namely w_1 , and w_2 , are fixed.

- Write the Lagrangian of the cost minimization problem of the firm.
- Calculate the first order conditions of the cost minimization problem.
- Calculate the conditional input factor demands.
- Calculate the cost function of the firm.

Hints

- The Lagrangian of the problem is obtained by subtracting from the objective of the firm the production constraint multiplied by the Lagrange multiplier λ . Specifically,

$$\mathcal{L} = w_1x_1 + w_2x_2 - \lambda \left(Ax_1^{\alpha}x_2^{(1-\alpha)r} - q \right).$$

- Differentiating the Lagrangian of the problem gives the first order conditions

$$\begin{aligned} w_1 &= \lambda A \alpha x_1^{\alpha-1} x_2^{(1-\alpha)r} = \lambda \alpha r \frac{q}{x_1}, \\ w_2 &= \lambda A (1-\alpha) r x_1^{\alpha} x_2^{(1-\alpha)r-1} = \lambda (1-\alpha) r \frac{q}{x_2}. \end{aligned}$$

- Solving the first order conditions for x_1 and x_2 , respectively, and substituting to the production function results in

$$q = A \left(\lambda \alpha r \frac{q}{w_1} \right)^{\alpha} \left(\lambda (1-\alpha) r \frac{q}{w_2} \right)^{(1-\alpha)r} = A \lambda^r r^r \left(\frac{\alpha}{w_1} \right)^{\alpha} \left(\frac{1-\alpha}{w_2} \right)^{(1-\alpha)r} q^r,$$

from which we obtain

$$\lambda = A^{-\frac{1}{r}} r^{-1} \left(\frac{\alpha}{w_1} \right)^{-\alpha} \left(\frac{1-\alpha}{w_2} \right)^{-(1-\alpha)} q^{\frac{1-r}{r}},$$

Substituting back to the first order conditions, we can eliminate λ , and solve for x_1 , x_2 to obtain the conditional demand functions

$$\begin{aligned} x_1 &= \lambda \alpha r \frac{q}{w_1} = \left(\frac{w_2}{w_1} \frac{\alpha}{1-\alpha} \right)^{1-\alpha} \left(\frac{q}{A} \right)^{\frac{1}{r}}, \\ x_2 &= \lambda (1-\alpha) r \frac{q}{w_2} = \left(\frac{w_1}{w_2} \frac{1-\alpha}{\alpha} \right)^{\alpha} \left(\frac{q}{A} \right)^{\frac{1}{r}}. \end{aligned}$$

- The easiest way to calculate the cost function of the firm is to use the first order conditions of part 2 and the objective function. We then have

$$\begin{aligned} c(q, w_1, w_2) &= w_1 \lambda \alpha r \frac{q}{w_1} + w_2 \lambda (1-\alpha) r \frac{q}{w_2} \\ &= \lambda r q \\ &= \left(\frac{\alpha}{w_1} \right)^{-\alpha} \left(\frac{1-\alpha}{w_2} \right)^{-(1-\alpha)} \left(\frac{q}{A} \right)^{\frac{1}{r}}. \end{aligned}$$

5. Cost Types

5.1. Group A

1. Consider the cost function $c(q) = 16 + 4q^2$. Calculate and plot

- the average cost function,
- the marginal cost function,
- the average variable cost function.
- the average fixed cost function.
- What is the level of output that yields the minimum average production cost?

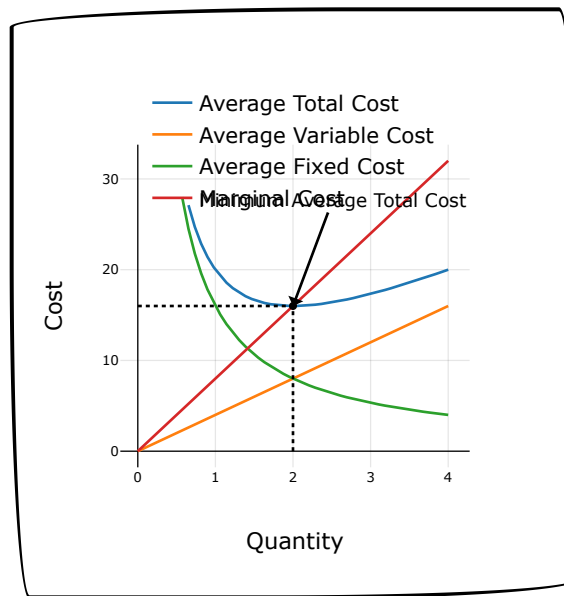
Hints

Cost type	Expression
Total cost	$c(q) = 16 + 4q^2$
Average cost	$\bar{c}(q) = \frac{16}{q} + 4q$
Average variable cost	$\bar{\mu}(q) = 4q$
Average fixed cost	$\bar{\sigma}(q) = \frac{16}{q}$
Marginal cost	$c'(q) = 8q$

The minimum average cost is obtained by the first order condition

$$-\frac{16}{q^2} + 4 = 0,$$

which implies that $q = 2$.



2. Consider the cost function obtained by a Cobb-Douglas production function

$$c(q; w_1, w_2) = \left(\frac{\alpha}{w_1}\right)^{-\alpha} \left(\frac{1-\alpha}{w_2}\right)^{-(1-\alpha)} \left(\frac{q}{A}\right)^{\frac{1}{r}},$$

and let $w_1 = w_2 = \alpha = 1/2$, $A = 1$, and $r > 0$. Show that the average cost function is

- increasing for decreasing returns to scale ($r < 1$),
- decreasing for increasing returns to scale ($r > 1$), and
- constant for constant returns to scale ($r = 1$).

Hints

The average cost function for the given parameters values is given by

$$\bar{c}(q) = q^{\frac{1-r}{r}},$$

with derivative

$$\bar{c}'(q) = \frac{1-r}{r} q^{\frac{1-2r}{r}}.$$

If $r < 1$, then $\bar{c}' > 0$ and $\bar{c}$ is increasing. If $r > 1$, then $\bar{c}' < 0$ and $\bar{c}$ is decreasing. Finally, if $r = 1$, then $\bar{c}' = 0$ and $\bar{c}$ is constant.

6. Firm Supply

6.1. Group A

1. Consider a price-taking, profit maximizing firm with cost function $c(q) = 1000 + 10q^2$.

- Calculate the supply function.
- Find the output level that minimizes the average cost.
- Calculate the profit function.

- d. What is the marginal effect of a price change on profit?
- e. What is the minimum price for which the firm makes non negative profit?

Hints

- a. The first order condition of the profit maximization problem is $20q = p$. Solving for q results in the supply function $s(p) = p/20$.
- b. The average cost is determined by

$$\bar{c}(q) = \frac{1000}{q} + 10q.$$

We can calculate the minimum average cost by solving the equation $\bar{c}(q) = c'(q)$, which is equivalent to

$$\frac{1000}{q} + 10q = 20q.$$

This results to $q = 10$.

- c. The profit function is given by

$$\begin{aligned}\pi(p) &= ps(p) - c(s(p)) \\ &= \frac{p^2}{20} - 1000 - 10 \frac{p^2}{400} \\ &= -1000 + \frac{p^2}{40}.\end{aligned}$$

- d. The effect of a marginal price change on profit is $\pi'(p) = p/20$.
- e. The firm makes non negative profit for prices such that $\pi(p) \geq 0$, which implies that $p \geq 200$.

2. Suppose that a firm has a supply function given by $s(p) = 100 + 4p$. Calculate the inverse supply function.

Hints

We solve for p to get

$$p(q) = -25 + \frac{q}{4}.$$

7. Market Supply

7.1. Group A

1. Suppose that a market consists of two firms, one with supply $s_1(p) = -10 + p$ and one with $s_2(p) = -15 + p$. Calculate the market supply function. At which prices does the market supply function have kinks?

Hints

Note that s_1 is valid for $p \geq 10$ because the resulting supplied quantities are negative for smaller prices. Similarly, s_2 is valid for $p \geq 15$. Therefore, the market supply is given by

$$\begin{aligned}s(p) &= s_1(p)\mathbb{1}_{[10,\infty)}(p) + s_2(p)\mathbb{1}_{[15,\infty)}(p) \\ &= \begin{cases} 0 & p < 10 \\ -10 + p & 10 \leq p < 15 \\ -25 + 2p & 15 \leq p. \end{cases}\end{aligned}$$

Thus, the market supply function has two kinks, one at $p = 10$ and one at $p = 15$.

2. Consider a market with multiple price-taking firms, all of which have cost functions $c(q) = q^2 + 1$ for $q \geq 0$. Suppose that entry is free and instantaneous in the market, the market demand function is $d(p) = 52 - p$, and the market perpetually clears.

- a. Calculate the supply and profit functions of an individual firm.
- b. Assume that n is a natural number (positive integer). What is the market supply function if there are n identical firms in the market?
- c. What does market-clearing imply for the relationship between the number of firms and the market price?
- d. Use the last condition to write the individual firm's profit as a function of the number of firms in the market.
- e. Find the equilibrium number of firms.
- f. What will be the equilibrium supplied quantity in the market?
- g. Suppose that demand shifts to $d(p) = 52.5 - p$. Calculate the equilibrium number of firms and the equilibrium supplied quantity. Are firms' profits zero in this case?

Hints

- a. The firm i 's profit maximization condition gives the inverse supply function

$$p = c'(q) = 2q,$$

from which we get the supply $s_i(p) = p/2$. One can then calculate the profit function of the firm as

$$\pi_i(p) = p \frac{p}{2} - \frac{p^2}{4} - 1 = \frac{p^2}{4} - 1.$$

b. With n identical firms, the market supply is

$$s(p) = \sum_{i=1}^n s_i(p) = \frac{n}{2}p.$$

c. Equating market demand and supply gives

$$\begin{aligned}\frac{n}{2}p &= s(p) = d(p) = 52 - p \iff \\ p &= \frac{104}{n+2}.\end{aligned}$$

d. The last equation implies that the firm's profit can be rewritten as

$$\pi_i(n) = \left(\frac{52}{n+2} \right)^2 - 1.$$

e. As long as entry is free and instantaneous, firms enter the market when they can make positive profits. Therefore, the equilibrium number of firms is exactly such that if one more firm enters the market, profits turn negative. Hence, the equilibrium number of firms is given by the greatest natural number n for which $\pi(n) \geq 0$. The non-negative profit condition is satisfied if

$$\frac{52}{n+2} \geq 1,$$

or, equivalently, if $n \leq 50$. Therefore, the equilibrium number of firms is $n = 50$.

f. Since $n = 50$, the equilibrium price is $p = 2$. Substituting the equilibrium number of firms and price in the market supply equation results in $s(2) = 50$.

g. With the shifted market demand, the market clearing condition implies that

$$p = \frac{105}{n+2},$$

hence, profits as a function of n become

$$\pi_i(n) = \left(\frac{52.5}{n+2} \right)^2 - 1.$$

This implies that $n \leq 50.5$ and, thus, the equilibrium number of firms remains unchanged equal to $n = 50$. One then can calculate the equilibrium price by

$$p = \frac{105}{52} > 2.$$

The supplied market quantity is

$$s = \frac{50}{2} \frac{105}{52} > 50$$

The equilibrium price increases, the supplied quantity also increases, and individual firm profits turn positive, i.e.

$$\pi_i(n) = \left(\frac{52.5}{52} \right)^2 - 1 > 0.$$

8. Monopoly

8.1. Group A

1. Consider a monopolist with cost function $c(q) = 2q$. Suppose that market demand is given by $d(p) = 100 - 2p$. Calculate the monopolist's price, quantity, and profit. In addition, calculate the deadweight loss.

Hints

The monopolist solves

$$\max_p \{pd(p) - c(d(p))\}.$$

The first order condition of the problem is

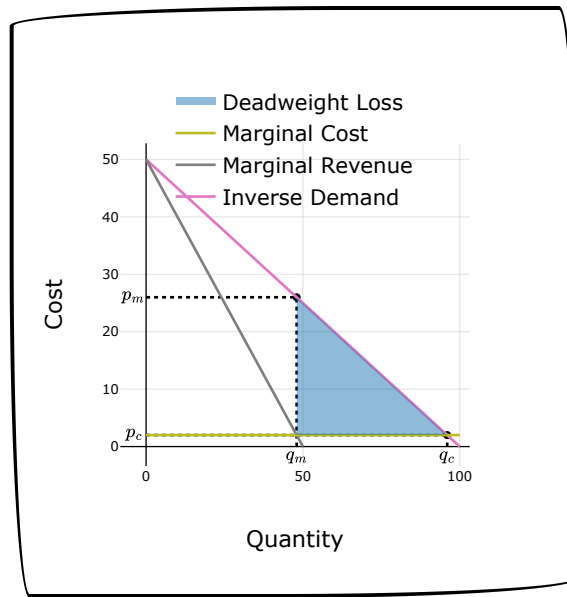
$$d(p) + pd'(p) - c'(d(p))d'(p) = 0.$$

Substituting the exercise's functional forms and calculating gives $p_m = 26$. Substituting back to demand gives $q_m = d(26) = 48$. The monopolist achieves profit

$$\pi(26) = 2d(26) - c(d(26)) = 26 \cdot 48 - 2 \cdot 48 = 1152.$$

When demand is affine, the deadweight loss can be calculated using a geometric argument and the formula for the area of a triangle. For this, we need the perfect competition price and quantity. The perfect competition price is given by $p_c = 2$ (why?) and, the perfect competition quantity is $q_c = 96$. One then calculates

$$D(q_c, q_m) = \frac{1}{2}(q_c - q_m)(p_m - p_c) = \frac{1}{2}(96 - 48)(26 - 2) = 576$$



2. Consider a monopolist with cost function $c(q) = q^2$. Suppose that market demand is given by $d(p) = 100/p$. Calculate the quantity that the monopolist would like to produce.

Hints

There are two differences compared to exercise 1. Firstly, demand is hyperbolic instead of affine, and secondly, the cost function is quadratic instead of linear. In this case, the maximization problem of the monopolist is

$$\max_p \left\{ p \frac{100}{p} - \frac{100^2}{p^2} \right\} = \max_p \left\{ 100 - \frac{100^2}{p^2} \right\}.$$

The derivative of the objective is $2 \cdot 10^4/p^3$, which is positive for all positive prices. This implies that the objective is increasing and, therefore, the maximization problem does not have a solution. The monopolist would like to let prices go to infinity. Taking this limit in the demand equation, we see that the monopolist would like to abstain from producing, i.e.,

$$\lim_{p \rightarrow \infty} d(p) = \lim_{p \rightarrow \infty} \frac{100}{p} = 0.$$

3. Consider a monopolist operating in a market where the commodity's demand depends on a qualitative attribute of the product besides its price. Specifically, demand is given by $d(s, p) = 3\sqrt{s}/p^3$, where s is the qualitative attribute of the market's commodity. The cost function of the monopolist also directly depends on the qualitative attribute, i.e., $c(s, q) = s/2 + 2q$. The monopolist's goal is to maximize profit by simultaneously choosing both the qualitative attribute level and the commodity's price. Calculate the monopolist price, the optimal level of the qualitative attribute, the output level, and the profit.

Hints

The problem of the monopolist is

$$\max_{s, p} \left\{ p d(s, p) - \frac{s}{2} - 2d(s, p) \right\} = \max_{s, p} \left\{ 3\sqrt{s} \left(\frac{1}{p^2} - 2 \frac{1}{p^3} \right) - \frac{s}{2} \right\}.$$

The first order conditions of the problem are

$$\begin{aligned} -2 \frac{1}{p^3} + 6 \frac{1}{p^4} &= 0, \\ \frac{3}{2} \frac{1}{\sqrt{s}} \left(\frac{1}{p^2} - 2 \frac{1}{p^3} \right) - \frac{1}{2} &= 0. \end{aligned}$$

The first condition involves only p , and the monopolistic price can be calculated from it. Solving for prices gives $p = 3$. By substituting p in the second condition, one can then solve for the qualitative attribute to get $s = 1/81$. Using the demand function, the output quantity level that the monopolist produces is $q = 1/81$. Finally, the profit is

$$\pi = 3\sqrt{\frac{1}{81}} \left(\frac{1}{9} - 2 \frac{1}{27} \right) - \frac{1}{2} \frac{1}{81} = \frac{1}{162}.$$

4. Can the monopolist select a price level for which demand is inelastic (i.e., $0 > e_d > -1$)?

Hints

No. By the monopolistic pricing condition, one can see that the monopolist would like to set a negative price in such a case. Specifically,

$$p = \frac{c'(d(p))e_d}{e_d + 1} < 0.$$

5. Suppose that the elasticity of demand $e_d < -1$ and the marginal production cost $c_1 > 0$ are constant. Calculate the marginal effect on prices of an improvement in the production technology reducing the marginal cost in a monopolistic market.

Hints

The monopolistic pricing condition is

$$p = \frac{c_1 e_d}{e_d + 1},$$

where elasticity does not depend on prices by the assumptions of the exercise. We can calculate

$$\frac{dp}{dc_1} = \frac{e_d}{e_d + 1}.$$

Since $e_d < -1$, then $dp/dc_1 > 0$. This implies that whenever a technological improvement reduces marginal cost, the monopolistic price in the market falls.

8.2. Group B

1. Consider a monopolist with cost function $c(q) = c_1 q$. Suppose that market demand is given by $d(p) = a/p^b$ for $a > 0$ and $b \geq 1$. Calculate the monopolist's price, quantity, and profit. In addition, calculate the deadweight loss for $c_1 = 2$, $a = 16$, and $b = 2$.

Hints

The difference with exercise 1 is that demand, in this case, is hyperbolic instead of affine. The first order condition of this problem yields

$$\frac{a}{p^b} + p \frac{-ab}{p^{b+1}} - c_1 \frac{-ab}{p^{b+1}} = 0,$$

which gives $p_m = c_1 b / (b - 1)$ and $q_m = a(b - 1)^b / (c_1 b)^b$. The monopolist makes profit

$$\pi_m = \frac{c_1 b}{b - 1} \frac{a(b - 1)^b}{(c_1 b)^b} - c_1 \frac{a(b - 1)^b}{(c_1 b)^b} = \frac{c_1 a(b - 1)^{b-1}}{(c_1 b)^b}.$$

The competition price is $p_c = c_1$ and the competition quantity is $q_c = a/c_1^b$. The deadweight loss can be calculated by

$$D(q_c, q_m) = \int_{q_m}^{q_c} (p(z) - c'(z)) dz,$$

where $p(q) = (a/q)^{1/b}$ is the inverse demand function. For the given parameter values, we get $q_c = 4$, $q_m = 1$, and

$$\begin{aligned} D(q_c, q_m) &= \int_{q_m}^{q_c} (a^{1/b} z^{-1/b} - c_1) dz \\ &= \int_1^4 (4z^{-1/2} - 2) dz \\ &= 8z^{1/2} - 2z \Big|_1^4 \\ &= 16 - 8 - (8 - 2) \\ &= 2 \end{aligned}$$

2. Show that if demand is affine and the marginal cost is constant, the rate of change of prices with respect to marginal cost in a monopolistic market is $1/2$.

Hints

Suppose that marginal cost is equal to c . The monopolistic pricing condition is

$$p = \frac{c e_d(p)}{e_d(p) + 1}.$$

Therefore, we can calculate

$$\begin{aligned} \frac{dp}{dc} &= \frac{e_d(p)}{e_d(p) + 1} + \frac{c}{(e_d(p) + 1)^2} \left(e'_d(p) \frac{dp}{dc} (e_d(p) + 1) - e_d(p) e'_d(p) \frac{dp}{dc} \right) \\ &= \frac{e_d(p)}{e_d(p) + 1} + \frac{c}{(e_d(p) + 1)^2} e'_d(p) \frac{dp}{dc}, \end{aligned}$$

from which we get

$$\frac{dp}{dc} = \frac{e_d(p)(e_d(p) + 1)}{(e_d(p) + 1)^2 - ce'_d(p)}.$$

Thus, we also need to calculate

$$\begin{aligned} e'_d(p) &= \frac{d\left(d'(p)\frac{p}{d(p)}\right)}{dp} \\ &= d''(p)\frac{p}{d(p)} + d'(p)\frac{1}{(d(p))^2}(d(p) - pd'(p)) \\ &= d''(p)\frac{p}{d(p)} + \frac{1}{p}d'(p)\frac{p}{d(p)}\left(1 - d'(p)\frac{p}{d(p)}\right) \\ &= d''(p)\frac{p}{d(p)} + \frac{1}{p}e_d(p)(1 - e_d(p)). \end{aligned}$$

Firstly, since demand is affine, we have $d'' = 0$ and, secondly, by the monopolist's pricing condition, we can rewrite the last expression as

$$e'_d(p) = \frac{e_d(p) + 1}{c}(1 - e_d(p)) = \frac{1 - (e_d(p))^2}{c}.$$

Substituting this result in the expression for dp/dc , we obtain

$$\frac{dp}{dc} = \frac{e_d(p)(e_d(p) + 1)}{(e_d(p) + 1)^2 - 1 + (e_d(p))^2} = \frac{1}{2}.$$

9. Monopoly Behavior

9.1. Group A

- Suppose that a monopolist can adopt a group pricing strategy in a market with two groups. Both groups' demands have constant price elasticities correspondingly equal to $e_{d,1}$ and $e_{d,2}$. The marginal production cost is constant at c_1 , and there are no fixed cost. What price does the monopolist charge to each group?

Hints

The monopolist solves

$$\max_{p_1, p_2} \{p_1 d_1(p_1) + p_2 d_2(p_2) - c_1 (d_1(p_1) + d_2(p_2))\}.$$

The optimality conditions for this problem are

$$d_i(p_i) + p_i d'_i(p_i) - c_1 d'_i(p_i) = 0 \quad (i = 1, 2).$$

From them, we deduce that

$$p_i = \frac{c}{\frac{1}{e_{d,i}} + 1} \quad (i = 1, 2).$$

10. Game Theory

10.1. Group A

- Consider the following game.

		Player B	
		Foo	Bar
Player A	Foo	2, 1	0, 0
	Bar	0, 0	1, 2

- Find all the pure strategy Nash equilibria.
- How do you interpret the objective of the players in this game?

Hints

- There are two Nash equilibria in this game, namely $\{Foo, Foo\}$ and $\{Bar, Bar\}$.

		Player B	
		Foo	Bar
Player A	Foo	2, 1	0, 0
	Bar	0, 0	1, 2

b. The objective of the players is to coordinate. The players would like to choose the same action. Games of this type are called coordination games.

2. Suppose that two players play a game, and it is known with certainty that player **A** is not choosing a Nash equilibrium strategy of a game. Should player **B** choose her Nash equilibrium strategy?

Hints

Not necessarily. In general, player **B** maximizes her payoff by choosing the best response strategy to player **A**'s strategy. If the Nash equilibrium strategy of player **B** is a dominant strategy, then she should choose it. Dominant strategies are best responses irrespective of the choice of other players. Therefore, if her Nash equilibrium strategy is a dominant strategy, she will still maximize her payoff by choosing it. For example, if player **A** chooses **Deny** in a prisoner's dilemma game, since the Nash equilibrium strategy of player **B**, namely **Confess**, is a dominant strategy, she maximizes her payoff by choosing it.

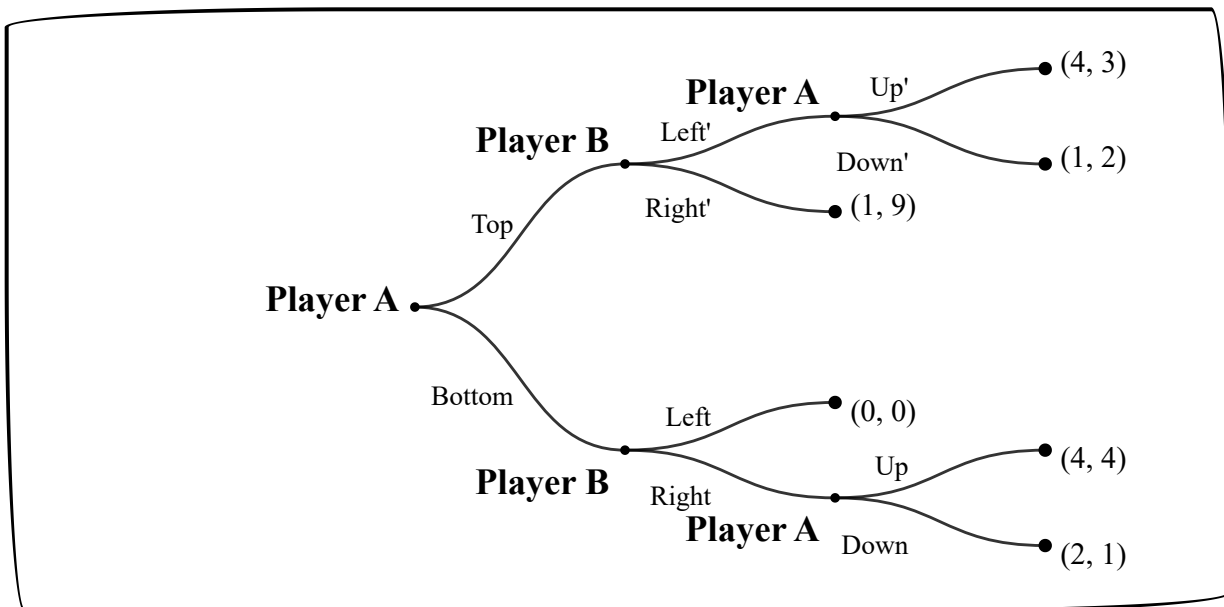
		Player B	
		Confess	Deny
Player A	Confess	-3, -3	0, -6
	Deny	-6, 0	-1, -1

In other games, in which her Nash equilibrium strategy is not dominant, the best response to the choice of player **A** can differ from the Nash equilibrium strategy and, therefore, player **B** might switch to a non Nash equilibrium strategy. For example, consider the game obtained by modifying the actions' labels and payoffs of the prisoner's dilemma in the following way.

		Player B	
		Left	Right
Player A	Top	-3, -3	0, -6
	Bottom	-6, 0	-1, <u>3</u>

In this game, if player **A** chooses **Bottom**, the best response of player **B** is **Right**. As is to be expected, **Left** is not a dominant strategy.

3. Find the equilibrium of the following game using backward induction.

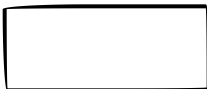


Hints

The Nash equilibrium obtained by backward induction is

$$\{(Bottom, Up', Up), (Right', Right)\}$$

4. Consider the game



Player B

		Left	Right
Player A	Top	3, 3	0, 1
	Bottom	1, 0	2, 2

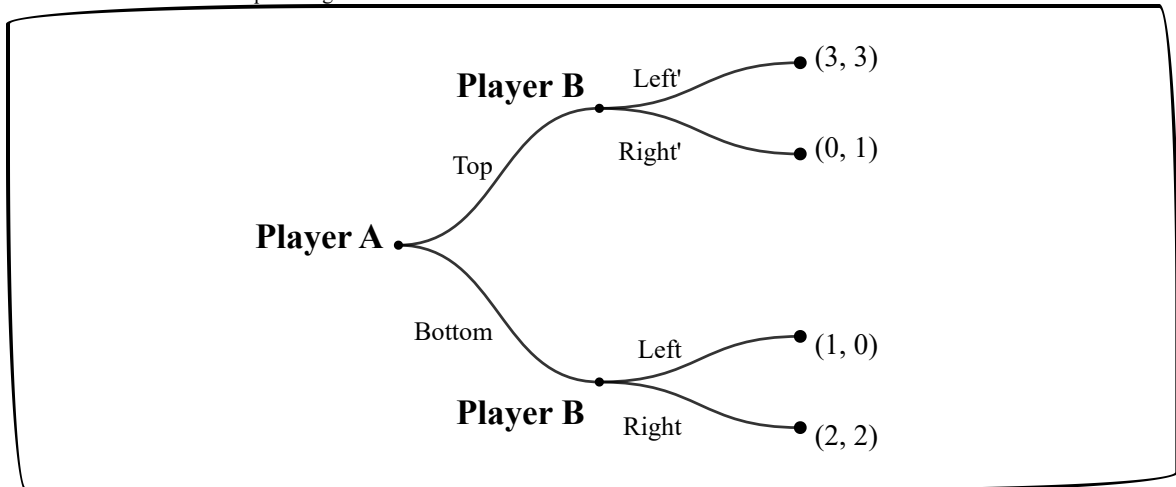
- Find all the pure strategy Nash equilibria of the game.
- Suppose that player A plays first. Draw the extensive form of the game.
- Solve the sequential game using backward induction.
- Are all Nash equilibria of the static game present in the sequential game? Why, or why not?

Hints

- There are two Nash equilibria given by $\{Top, Left\}$ and $\{Bottom, Right\}$.

		Player B	
		Left	Right
Player A	Top	3, 3	0, 1
	Bottom	1, 0	2, 2

- The extensive form of the sequential game is



- The Nash equilibrium obtained by backward induction is $\{Top, (Left', Right)\}$.
- Nash equilibria in which player A plays **Bottom** cannot be obtained by backward induction in the sequential game. This happens because player A prefers choosing **Top** at date 1 as she receives a higher payoff. Even if player B attempts to convince player A to choose **Bottom** by threatening that she will play $(Right', Right)$, her statement is not credible. Once player A chooses **Top**, player B receives a greater payoff by choosing **Left'**. In general, dynamic aspects in a game can affect the resulting credibility of some Nash equilibria. Therefore, when analyzing strategic interactions for which time is a central component, it is important not to neglect it because doing so might lead to less refined equilibrium outcomes.

10.2. Group B

- Consider the game

		Player B	
		Left	Right
Player A	Top	0, 2	2, 0
	Bottom	2, 0	0, 2

- Does the game have any pure strategy Nash equilibria?
- Find all the Nash equilibria of the game.

Hints

- No. There is no set of strategies that constitute best responses to one another.

		Player B	
		Left	Right
Player A	Top	0, <u>2</u>	<u>2</u> , 0
	Bottom	<u>2</u> , 0	0, <u>2</u>

- Suppose that player A randomizes by playing **Top** with probability $p > 0$ and **Bottom** with probability $1 - p$. If this mixed strategy is a best response, it must equalize the expected payoff of the strategies of player B because otherwise, player B would choose either **Left** or **Right**. The unique best response of player A to **Left** is **Bottom**, which means that the mixed strategy $(p, 1 - p)$ with $p > 0$ cannot be the best response to player B's choice of **Left**. Similarly, we can exclude the case of player B choosing **Right**.

Therefore, for the mixed strategy to be a best response, it should hold

$$\underbrace{p \cdot 2 + (1-p) \cdot 0}_{\text{Player B's expected payoff for Left}} = \underbrace{p \cdot 0 + (1-p) \cdot 2}_{\text{Player B's expected payoff for Right}}.$$

This implies that $p = 1/2$.

Suppose that player B randomizes by playing **Left** with probability $q > 0$ and **Right** with probability $1 - q$. Using similar arguments as above, the randomization of player B should equalize the payoff of the strategies of player A, namely

$$\underbrace{q \cdot 0 + (1-q) \cdot 2}_{\text{Player A's expected payoff for Top}} = \underbrace{q \cdot 2 + (1-q) \cdot 0}_{\text{Player A's expected payoff for Bottom}}.$$

The last condition implies that $q = 1/2$.

Combining the above, the unique Nash equilibrium of the game is given by the mixed strategies

$$\left\{ \left(p = \frac{1}{2}, 1-p = \frac{1}{2} \right), \left(q = \frac{1}{2}, 1-q = \frac{1}{2} \right) \right\}.$$

2. We consider the prisoner's dilemma as the stage game of a repeated game.

- Suppose that the stage game is repeated twice. Find the unique Nash equilibrium of the game.
- Suppose that the stage game is repeated an arbitrary finite number of times (say $n = 10^{10}$). Find the unique Nash equilibrium of the game.
- Why is it impossible for the players to coordinate and play the Pareto efficient strategies in the above cases?

Hints

- We can find the Nash equilibrium by backward induction. At date **2**, the unique Nash equilibrium of the stage game is obtained when both players choose **Confess**. Since there are no further dates in which the stage game is repeated, the players cannot coordinate in the Pareto efficient outcome by promising cooperation in the future. At date **1**, the unique Nash equilibrium of the stage game is obtained once more for both players choosing **Confess**. Since the only Nash equilibrium of the game at date **2** is the pair **{Confess, Confess}**, players cannot coordinate at the Pareto efficient outcome at date **1** by threatening to punish non-cooperation in the future. Therefore, the unique Nash equilibrium of the game is obtained by **{(Confess, Confess), (Confess, Confess)}**.
- If the game is repeated a finite number of times, the unique Nash equilibrium is obtained when both players choose **Confess** at every date. This result can be obtained by backward induction. At the final date of the game, the unique Nash equilibrium of the stage game is **{Confess, Confess}**, justified as in the case of the two repetitions. Suppose that we have established that the unique Nash equilibrium of the last k repetitions of the game is

$$\left\{ \underbrace{(\text{Confess}, \dots, \text{Confess})}_{k\text{-times}}, \underbrace{(\text{Confess}, \dots, \text{Confess})}_{k\text{-times}} \right\}.$$

With the argument used for date 1 in the case of two repetitions, one concludes that the unique Nash equilibrium of the $(n - k)$ -th date of the game is **{Confess, Confess}**. Continuing in this respect, we conclude that the unique Nash equilibrium of the repeated game is obtained when both players choose **Confess** at every date.

- When a stage game with a unique Nash equilibrium is repeated a finite number of times, the players cannot establish credible promises or threats. However, if the repeated game has an infinite horizon, then coordination strategies that deviate from stage game Nash equilibria are feasible, even if there is only a unique Nash equilibrium in the stage game. An example of this is the infinitely repeated prisoner's dilemma. Promises and threats can be established when a stage game is repeated only a finite number of times, only if the stage game has multiple Nash equilibria, one of which can be used in coordination and one in non-coordination.

10.3. Group C

1. Consider the prisoner's dilemma game in which if both players do not confess, they get a payoff of **-1**; if they both confess, they get a payoff of **-6**; and if one confesses and the other does not, the one that confessed receives a payoff of **0** and the other a payoff of **-10**

- Classify the strategy profiles into those yielding Pareto efficient and Pareto inefficient allocations.
- Show that for each strategy profile yielding a Pareto efficient allocation, there exist positive weights $(\alpha, 1 - \alpha)$ such that the weighted average of the payoffs of the players for this profile is greater or equal to the weighted average payoffs of all other strategy profiles.
- Show that for each strategy profile yielding Pareto inefficient allocations, such weights do not exist.

Hints

- The game has three Pareto efficient allocations. One from the strategy profile for which both players do not confess, and both combinations where one player confesses and the other does not. The only Pareto inefficient allocation is the result of the strategy profile in which both players confess since the profile in which both players do not confess constitutes a Pareto improvement.
- Let u_i denote the payoff of player $i \in \{A, B\}$. We consider the cases for the strategy profiles **{N, N}**, **{N, C}**, and **{C, N}**, where **C** is short for the strategy confess and **N** for the strategy not confess.

For the case **{N, N}**, choose $\alpha = 1/2$ and let

$$W_1(s_A, s_B) = \frac{1}{2}u_A(s_A, s_B) + \frac{1}{2}u_B(s_A, s_B).$$

Then, we have

$$\underbrace{W_1(N, N)}_{=-1} > \underbrace{W_1(C, N)}_{=-5} = \underbrace{W_1(N, C)}_{=-5} > \underbrace{W_1(C, C)}_{=-6}.$$

For the case **{N, C}**, choose $\alpha = 1/20$ (any $\alpha \leq 1/10$ works) and let

$$W_2(s_A, s_B) = \frac{1}{20}u_A(s_A, s_B) + \frac{19}{20}u_B(s_A, s_B).$$

Then, we have

$$\underbrace{W_2(N, C)}_{=-1/2} > \underbrace{W_2(N, N)}_{=-1} > \underbrace{W_2(C, C)}_{=-6} > \underbrace{W_2(C, N)}_{=-9.5}.$$

The last case $\{C, N\}$ is symmetric to the case $\{N, C\}$ and we can choose any $\alpha \in [9/10, 1]$ to show the claim.
c. Let α take an arbitrary value in $(0, 1)$ and define

$$W_3(s_A, s_B) = \alpha u_A(s_A, s_B) + (1 - \alpha) u_B(s_A, s_B).$$

We can then calculate

$$W_3(C, C) = -6 < -1 = W_3(N, N).$$

Since the choice of α was arbitrary, the claim follows.

11. Oligopoly

11.1. Group A

- Suppose that there are two firms in a market with an affine inverse demand function $p(q) = p_0 + p_1 q$, where $p_0 > 0$ and $p_1 < 0$. The firms compete by simultaneously choosing quantities and have constant marginal costs equal to c_1 . Production has no fixed cost. Find the equilibrium outputs, price, and profits.

Hints

Let $i \neq j$ for $i, j = 1, 2$. Firm i chooses its strategy by solving

$$\max_{q_i} \{p(q_i + q_j)q_i - c_1 q_i\}.$$

For interior solutions, the firm sets

$$p_0 + 2p_1 q_i + p_1 q_j - c_1 = 0,$$

which gives the best response function

$$b_i(q_j) = \frac{c_1 - p_0 - p_1 q_j}{2p_1}.$$

Since the problem is symmetric, we can set $q_i = q_j$ in the first order condition to easily obtain the equilibrium output

$$q_i = \frac{c_1 - p_0}{3p_1}.$$

Thus, the total market quantity is

$$q = q_1 + q_2 = 2 \frac{c_1 - p_0}{3p_1},$$

which implies that the equilibrium price is

$$p(q) = \frac{p_0 + 2c_1}{3}.$$

Finally, each firm i makes profit

$$\pi_i = (p(q) - c_1)q_i = -\frac{(c_1 - p_0)^2}{9p_1}.$$

- Can an oligopolistic market structure result in an efficient level of output?

Hints

Yes. The simultaneous price competition with homogeneous products leads firms to choose the competitive price, which, in turn, results in the production of the efficient output level. In contrast, competition in quantities results in deadweight losses.

- Consider a market with inverse demand function $p(q) = 100 - 2q$. The total cost function for any firm in the market is given by $c(q) = 4q$.

- What is the marginal cost for any firm in the market?
- Calculate the perfect competition market quantity and price.
- Suppose that the market consists of two firms that compete by simultaneously choosing their supplied quantities. Find the best responses and the equilibrium market output and price?
- Draw the two best responses in a single graph to indicate the equilibrium point.
- Suppose that the two firms collude by maximizing and splitting their joint profit. Find market output and the market price. Compare them with the non-collusive market output and price.

- f. Suppose that a firm decides to deviate from the collusive output, while the other one keeps its collusive strategy unchanged. What are the firms' output levels and profits?
g. Is the collusion strategy sustainable in this game?

Hints

a. $c'(q) = 4$

- b. The perfect competition price is equal to the marginal production cost, i.e., $p_c = 4$. By inverting the inverse demand and substituting the competitive price, we obtain

$$q_c = \frac{100 - p_c}{2} = 48.$$

- c. Each firm solves

$$\max_{q_i} \{p(q_i + q_j)q_i - c(q_i)\},$$

with first order condition

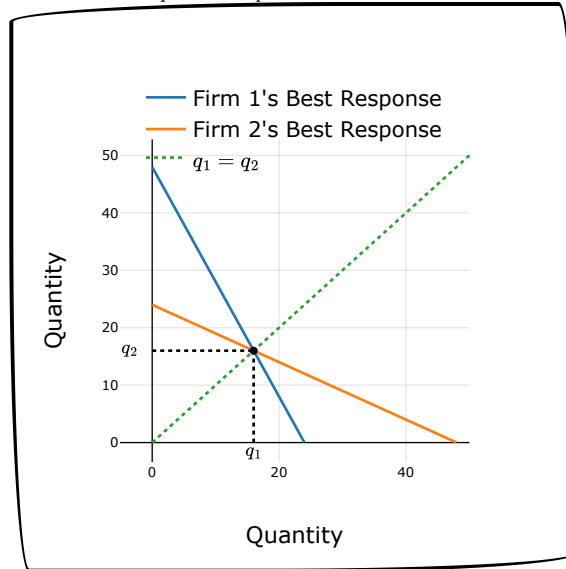
$$100 - 4q_i - 2q_j - 4 = 0.$$

Therefore, the best response of each firm is

$$b_i(q_j) = \frac{48 - q_j}{2}$$

and the firm's optimal quantity is $q_i = 16$. The market quantity is $q_s = 32$, and the market price is $p_s = 36$.

- d. The intersection of the best responses indicates the equilibrium quantities.



- e. In this case, the two firms solve the monopolist problem and split the profit. The monopolistic quantity is given by

$$q_m = \frac{100 - 4}{4} = 24,$$

in which case, the market price is $p_m = 52$. With collusion, the market power of the firms increases, the produced output level decreases ($q_m = 24 < 32 = q_s$), while the market price increases ($p_m = 52 > 36 = p_s$).

- f. Suppose that firm i deviates from collusion, while firm j keeps producing half of the monopolistic quantity. The best deviation of firm i can be calculated from its best response function, namely

$$q_b = b_i\left(\frac{q_m}{2}\right) = \frac{48 - 12}{2} = 18.$$

The best deviation for the firm is to produce at an output level greater than the level of the simultaneous quantity competition (i.e., $q_b = 18 > 16 = q_s/2$). Firm i takes advantage of the fact that firm j is not producing at its best response output level and produces more to grab a greater share of the market. The market quantity is

$$q_w = q_b + \frac{q_m}{2} = 18 + 12 = 30.$$

Then, the price is $p_w = 40$, and the firms' profits are

$$\begin{aligned} \pi_i &= (p_w - c)q_b = 36 \cdot 18 = 648 \\ &> 576 = 48 \cdot 12 = (p_m - c) \frac{q_m}{2} = \frac{\pi_m}{2} \end{aligned}$$

and

$$\pi_j = (p_w - c) \frac{q_m}{2} = 36 \cdot 12 = 432 < 576 = \frac{\pi_m}{2}.$$

g. No. Each firm has a profitable deviation from producing half of the monopolistic output in this static game. More than a single date is needed for firms to be able to collude successfully.

11.2. Group B

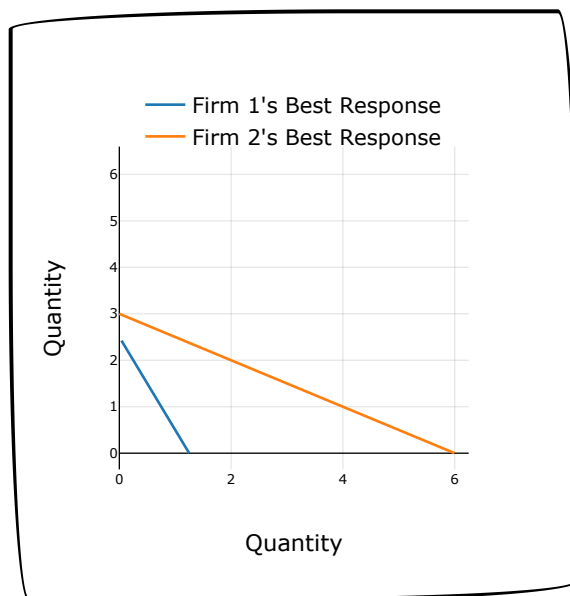
1. Give an example in which there is no equilibrium in a duopoly market in which firms simultaneously compete using quantities. Draw the best responses of the firms in this market.

Hints

Suppose that the inverse demand function of the market is given by $p(q) = p_0 + p_1 q = 25 - 4q$. Let the marginal cost of firm 1 be $c_1 = 15$ and that of firm 2 to be $c_2 = 1$. For these values, if there was an equilibrium in this market, according to the best response functions, firm 1 should produce

$$q_1 = \frac{2c_1 - c_2 - p_0}{3p_1} = -\frac{30 - 1 - 25}{12} < 0.$$

Therefore, this market does not have an equilibrium. Visually, this results in two best response functions that do not cross each other.



2. Suppose there are n identical firms in a market competing by simultaneously choosing quantities. Show that the elasticity of the market demand must be less than $-1/n$ for an equilibrium to exist.

Hints

This result is the analog of the upper bound for the elasticity of monopolistic markets discussed in exercise 4. In particular, the exact result of the monopolistic case is obtained for $n = 1$. To simplify notation, let $q = \sum_{j=1}^n q_j$. Each firm solves

$$\max_{q_i} \{p(q)q_i - c(q_i)\},$$

with the first order condition of firm i being

$$p'(q)q_i + p(q) - c'(q_i) = 0.$$

By the inverse function theorem, we can rewrite the above condition as

$$0 = \frac{1}{d'(p(q))} q_i + p(q) - c'(q_i),$$

where d is the demand function, and, then, express it in terms of the demand elasticity as

$$\frac{c'(q_i)}{p(q)} = \frac{1}{d'(p(q))} \frac{q}{p(q)} \frac{q_i}{q} + 1 = \frac{1}{e_d(p(q))} \frac{q_i}{q} + 1.$$

Since all firms are identical, the equilibrium, if it exists, is symmetric. This means that $q_i = q/n$. Substituting the symmetry condition in the first order condition gives

$$p(q) = \frac{c'(q_i)}{\frac{1}{e_d(p(q))} \frac{1}{n} + 1}.$$

Since the marginal cost is positive, positive prices require that

$$\frac{1}{e_d(p(q))} \frac{1}{n} + 1 > 0,$$

which is equivalent to $e_d(p(q)) < -1/n$.

12. Externalities

12.1. Group A

1. Consider two monopolistic markets intermingled by an externality. In the first market, the firm's profit as a function of output is $\pi_1(q_1) = 48q_1 - q_1^2$. The output of the first monopolist affects the profit of the monopolist in the second market as an externality. The second firm's profit as a function output is $\pi_2 = (60 - q_1)q_2 - q_2^2$.

- Suppose that each firm independently maximizes its profit. Calculate the optimal quantity and profit of each market. Moreover, calculate the total profit for both markets.
- Suppose that the firm in market **1** is not allowed to produce any output. Calculate the optimal quantity and profit in market **2**.
- Suppose that firm **1** has to pay a transfer to firm **2**, equal to the damages caused by the production externality. How do the profit functions of the two firms change? Calculate the optimal quantity and profit of each market. Calculate also the total profit in both markets.
- Suppose that the two firms merge. What is the resulting profit function of the merged firm? Calculate the optimal quantity and profit.
- Compare the profits of the above cases. Which case is the most efficient?

Hints

- a. Firm 1 solves

$$\max_{q_1} \{48q_1 - q_1^2\},$$

which results in $q_1 = 24$. Firm 2 solves

$$\max_{q_2} \{(60 - q_1)q_2 - q_2^2\},$$

which gives

$$q_2 = 30 - \frac{q_1}{2} = 30 - 12 = 18.$$

Then, we can calculate the profits

$$\begin{aligned}\pi_1 &= 48 \cdot 24 - 24^2 = 576, \\ \pi_2 &= (60 - 24)18 - 18^2 = 324, \\ \pi &= \pi_1 + \pi_2 = 900.\end{aligned}$$

- b. In this case, $q_1 = 0$ and the quantity that maximizes profit in the second market is

$$q_2 = 30 - \frac{q_1}{2} = 30.$$

The firm's profit is

$$\pi = \pi_2 = (60 - 0)30 - 30^2 = 900.$$

- c. When firm **1** produces q_1 , the damage caused by the externality to firm **2** is equal to q_1q_2 . With the transfer, firm 1 solves

$$\max_{q_1} \{48q_1 - q_1^2 - q_1q_2\},$$

which essentially moves the externality to the first market. This results in the necessary condition

$$q_1 = 24 - \frac{q_2}{2}.$$

Firm **2** solves

$$\max_{q_2} \{60q_2 - q_2^2\},$$

which gives $q_2 = 30$. Substituting into the best response of the first firm, we get $q_1 = 9$. Then, the profits are given by

$$\begin{aligned}\pi_1 &= (48 - 30) \cdot 9 - 9^2 = 81, \\ \pi_2 &= 60 \cdot 30 - 30^2 = 900, \\ \pi &= \pi_1 + \pi_2 = 981.\end{aligned}$$

- d. The merged firm maximizes

$$\max_{q_1, q_2} \{48q_1 - q_1^2 + 60q_2 - q_2^2 - q_1q_2\}.$$

The first order conditions of the problem are

$$\begin{aligned}48 - 2q_1 - q_2 &= 0, \\ 60 - 2q_2 - q_1 &= 0.\end{aligned}$$

Solving the above system gives $q_1 = 12$ and $q_2 = 24$. The profit of the merged firm is

$$\pi = 48 \cdot 12 - 12^2 + 60 \cdot 24 - 24^2 - 12 \cdot 24 = 1008.$$

e. The merged case is the Pareto efficient structure because the externality is internalized.

Case	Structure	π_1	π_2	π
1	Externality in the second market	576	324	900
2	First market shuts down	0	900	900
3	Externality in the first market	81	900	981
4	Internalized externality			1008

13. Public Goods

13.1. Group A

1. Suppose there are two individuals, **1** and **2**, each consuming one private good in the amounts x_1 and x_2 , respectively. The price of this private good is p . In addition, they consume a public good G . The marginal cost of production of G is equal to one. The amount of the public good is determined by $G = g_1 + g_2$, where g_1 and g_2 are the individual contributions of the players. Individual i 's preferences are represented by a Cobb-Douglas utility function $u(x_i, G) = x_i^\alpha G^\beta$. Both players have a budget of B .

- Explain why in this case, G can be considered a public good.
- Set up individual i 's budget constraint.
- Find individual i 's best response contribution g_i .
- Compute the Nash equilibrium for the contributions g_1 and g_2 .
- Does the Nash equilibrium allocation entail more or fewer resources directed to the public good than the Pareto efficient allocation?
- Argue that the Nash equilibrium is not Pareto-efficient.

Hints

- The good G is non-rivalrous, as both players consume the exact same good G , and non-excludable as the good is not priced in the market.
- The budget of individual i is $px_i + g_i \leq B$.
- Individual i 's optimization problem is

$$\max_{x_i, g_i} \{u(x_i, G) + \lambda(B - px_i - g_i)\},$$

where λ is the Lagrange multiplier. The first order conditions are

$$\begin{aligned} \alpha \frac{u(x_i, G)}{x_i} &= \lambda p, \\ \beta \frac{u(x_i, G)}{g_i} &= \lambda, \end{aligned}$$

which imply

$$\frac{\alpha}{\beta} G = px_i.$$

Combining the last condition with the budget constant, we get

$$\frac{\alpha}{\beta} G = B - g_i,$$

from which we conclude that the best response is

$$g_i = \frac{\beta}{\alpha + \beta} B - \frac{\alpha}{\alpha + \beta} g_j.$$

d. Due to symmetry, the Nash equilibrium can be easily calculated by setting $g_i = g_j$, which results in

$$g_n = \frac{\beta}{2\alpha + \beta} B.$$

e. The Pareto efficient solutions are obtained by solving the joint maximization problem

$$\max_{x_1, x_2, G} \{u(x_1, G) + u(x_2, G) + \mu(2B - p(x_1 + x_2) - G)\},$$

where μ is the Lagrange multiplier of this problem. The first order conditions are

$$\begin{aligned} \alpha \frac{u(x_1, G)}{x_1} &= \mu p, \\ \alpha \frac{u(x_2, G)}{x_2} &= \mu p, \\ \beta \frac{u(x_1, G) + u(x_2, G)}{G} &= \mu. \end{aligned}$$

The first two conditions and the budget constraint imply

$$\alpha (u(x_1, G) + u(x_2, G)) = \mu p (x_1 + x_2) = \mu (2B - G).$$

Combining with the third first order condition gives

$$G_p = \frac{\beta}{\alpha + \beta} 2B.$$

For $\alpha \neq 0$, $G_p \neq 2g_n$, from which we conclude that the Nash equilibrium is not Pareto efficient.

f. In the Nash equilibrium, fewer than the Pareto efficient resources are allocated in the production of the public good. Specifically, for all $\alpha > 0$, we have $G_p > 2g_n$.

13.2. Group B

1. Consider a public good game with two players whose payoffs are determined by $\pi_1(q) = (1 + q)(B_1 - qx_1)$ and $\pi_2(q) = (2 + q)(B_2 - qx_2)$. In these expressions, x_1 and x_2 represent the player specific, fixed contributions to producing the public good, and q the public good output. Players have a binary choice concerning the public good; they can commonly choose $q = 1$ and purchase the public good, or $q = 0$ and abstain from it. Player 1 has budget B_1 , while player 2 has budget B_2 .

- What is the maximum amount that player 1 is willing to contribute to the public good?
- What is the maximum amount that player 2 is willing to contribute to the public good?
- Suppose that $B_1 = 100$ and $B_2 = 75$. By producing the public good, the players have a Pareto improvement over not producing it, as long as the production cost of the public good is no greater than $\hat{x}$. Find $\hat{x}$.
- Find a combination of specific contributions x_1 and x_2 whose sum is less than $\hat{x}$, but for which letting the players individually maximize their payoffs does not result in a Pareto efficient allocation.

Hints

- a. Player 1 prefers consuming the public good if and only if

$$\pi_1(1) \geq \pi_1(0) \iff 2(B_1 - x_1) \geq B_1 \iff x_1 \leq \frac{B_1}{2}.$$

- b. For player 2, an analogous approach gives

$$\pi_2(1) \geq \pi_2(0) \iff 3(B_2 - x_2) \geq 2B_2 \iff x_2 \leq \frac{B_2}{3}.$$

- c. The total production cost of the public good, say x , has to be covered by the sum of the individual contributions, i.e., $x_1 + x_2 \geq x$. Both players prefer covering the cost if

$$\begin{aligned} x &\leq x_1 + x_2 \\ &\leq \frac{B_1}{2} + \frac{B_2}{3} \\ &\leq \frac{3B_1 + 2B_2}{6} \\ &= \frac{300 + 150}{6} = 75 =: \hat{x} \end{aligned}$$

- d. Suppose that $x_2 = 50$ and $x_1 = 20$. We then have $x_1 + x_2 = 70 < 75 = \hat{x}$, but player 2 prefers not to contribute to the public good production because $x_2 > 25$.
-