

Strategic Interactions in Markets

Table of Contents

[1. Markets, Games, and Competition](#)

[1.1. Group A](#)

[2. Simultaneous Games](#)

[2.1. Group A](#)

[2.2. Group B](#)

[3. Simultaneous Competition](#)

[3.1. Group A](#)

[3.2. Group B](#)

[4. Dynamic Games](#)

[4.1. Group A](#)

[4.2. Group B](#)

[5. Dynamic Competition](#)

[5.1. Group A](#)

[5.2. Group B](#)

[6. Collusion and Market Power](#)

[6.1. Group A](#)

[6.2. Group B](#)

1. Markets, Games, and Competition

1.1. Group A

1. Consider a market with inverse demand given by $p(q) = p_0 + p_1 q$ with $p_0 > 0$ and $p_1 < 0$.
 - a. Calculate the demand function (i.e., invert p). For which values of price is demand non-negative?
 - b. Calculate the demand elasticity as a function of price using the demand function. What is the sign of elasticity?
 - c. Calculate the demand elasticity as a function of quantity using the inverse demand function.
 - d. Substitute the quantity variable in the result of part (c) with the demand calculated in part (a) and verify that you get the result of part (b).
 - e. Calculate the quantity for which demand exhibits unitary elasticity.

Hints

- a. Fix a value for p and solve for q to get

$$q(p) = -\frac{p_0}{p_1} + \frac{1}{p_1} p.$$

Since $p_1 < 0$, demand is non-negative if and only if $p \leq p_0$.

- b. We calculate

$$\begin{aligned}
E(p) &= \frac{dq}{dp} \frac{p}{q(p)} \\
&= q'(p) \frac{p}{q(p)} \\
&= \frac{1}{p_1 - \frac{p_0}{p_1} + \frac{1}{p_1} p} p \\
&= \frac{p}{-p_0 + p}
\end{aligned}$$

For every $0 \leq p \leq p_0$ is non-positive.

c. We have

$$\begin{aligned}
E(q) &= \frac{dq}{dp} \frac{p(q)}{q} \\
&= \frac{1}{\frac{dp}{dq}} \frac{p(q)}{q} \\
&= \frac{1}{p'(q)} \frac{p(q)}{q} \\
&= \frac{1}{p_1} \frac{p_0 + p_1 q}{q}
\end{aligned}$$

d. Substituting gives

$$\begin{aligned}
E(q(p)) &= \frac{p_0 + p_1 q(p)}{p_1 q(p)} \\
&= \frac{p_0 - p_0 + p}{-p_0 + p} \\
&= \frac{p}{-p_0 + p}
\end{aligned}$$

e. Using part (c), we have $E(q) = -1$ if and only if

$$p_0 + p_1 q = -p_1 q,$$

or equivalently

$$q = -\frac{p_0}{2p_1}.$$

2. Consider a monopolist in a market with inverse demand $p(q) = p_0 + p_1 q$, where $p_0 > 0$ and $p_1 < 0$. The monopolist's cost function is given by $c(q) = c_1 q$, where $0 < c_1$.

- Set up the maximization problem of the monopolist for which the monopolist chooses the produced quantity.
- Calculate the monopoly quantity, price, and profit.
- Instead, suppose that the monopolist is choosing prices. Calculate the demand function and set up the maximization problem.
- Calculate the quantity, price, and profit. Compare them to the maximization problem over quantities. Are the results the same? Why, or why not?
- Calculate the consumer welfare and the total welfare.
- Compare with the welfare of the perfect competition case and calculate the deadweight loss.

Hints

a. The monopolist solves

$$\max_q \{p(q)q - c(q)\}$$

b. The first order condition is

$$p'(q)q + p(q) = c'(q) \iff p_0 + 2p_1q = c_1,$$

which gives

$$q_m = -\frac{p_0 - c_1}{2p_1},$$
$$p_m = \frac{c_1 + p_0}{2},$$

and

$$\pi_m = -\frac{(p_0 - c_1)^2}{4p_1}.$$

c. The demand function is

$$q(p) = -\frac{1}{p_1}(p_0 - p)$$

and the monopolist problem is

$$\max_p \{q(p)p - c(q(p))\}.$$

d. The first order condition is

$$q'(p)p + q(p) = c'(q)q'(p) \iff \frac{1}{p_1}(2p - p_0) = \frac{c_1}{p_1},$$

which leads to the same price, quantity, and profit as in part (b). The result is the same because inverse demand is affine (more generally because inverse demand is one-to-one).

e. The consumer welfare is

$$C_m = \int_0^{q_m} (p(q) - p_m) dq$$
$$= p_0 q_m + \frac{p_1}{2} q_m^2 - p_m q_m$$
$$= \frac{1}{2} q_m (p_0 - p_m)$$
$$= -\frac{(p_0 - c_1)^2}{8p_1},$$

so the total welfare is

$$W_m = C_m + \pi_m = -3 \frac{(p_0 - c_1)^2}{8p_1}.$$

f. In perfect competition, $p_c = c_1$. Then,

$$q_c = \frac{c_1 - p_0}{p_1}$$

and profits are zero. The competition's total welfare is

$$\begin{aligned} W_c &= C_c \\ &= \int_0^{q_c} (p(q) - c_1) dq \\ &= p_0 q_c + \frac{p_1}{2} q_c^2 - c_1 q_c \\ &= \frac{1}{2} q_c (p_0 - p_c) \\ &= -\frac{(p_0 - c_1)^2}{2p_1}. \end{aligned}$$

Finally, the deadweight loss is

$$D = W_c - W_m = -\frac{(p_0 - c_1)^2}{8p_1}.$$

2. Simultaneous Games

2.1. Group A

- Consider the Bach or Stravinsky game (also known as Battle of the Sexes, which is outdated in current norms). A couple wishes to go to a music concert by either Bach or Stravinsky. One partner prefers Bach and the other Stravinsky. Nevertheless, their primary concern is to go out together.

		Player B	
		Bach	Stravinsky
Player A	Bach	2, 1	0, 0
	Stravinsky	0, 0	1, 2

- Find all the Nash equilibria.
- How do you interpret the objective of the players in this game?

Hints

- There are two pure Nash equilibria in this game, namely *(Bach, Bach)* and *(Stravinsky, Stravinsky)*.

		Player B	
		Bach	Stravinsky
Player A	Bach	<u>2</u> , <u>1</u>	0, 0
	Stravinsky	0, 0	<u>1</u> , <u>2</u>

There is also a mixed strategy equilibrium in which player *A* chooses *Bach* with $p = \frac{2}{3}$ and *Stravinsky* with $p = \frac{1}{3}$, while player *B* chooses *Bach* with $q = \frac{1}{3}$ and *Stravinsky* with $q = \frac{2}{3}$.

- b. The objective of the players is to coordinate. The players would like to choose the same action. Games of this type are called coordination games.
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2.2. Group B

1. Consider the Hawk and Dove game. Two animals fight over a food source with a value of 1. Each animal can behave aggressively (like a Hawk) or peacefully (like a Dove). If both animals behave peacefully, they split the food. If only one animal behaves aggressively, it gets all the food. If both animals act aggressively, the value of the food source is reduced by $c > 0$ (lost in the fight), and the animals split the remainder equally.

 - a. Write the normal form of the game.
 - b. Find all the Nash equilibria for every value of the cost parameter c .
 - c. How do you interpret the equilibria of each case?

Hints

- a. The normal form of the game is as follows.

		Animal B	
		Dove	Hawk
Animal A	Dove	1/2, 1/2	0, 1
	Hawk	1, 0	(1 - c)/2, (1 - c)/2

- b. We distinguish three cases based on the value of the parameter c . For $c < 1$, the game has two pure strategy Nash equilibria. These are $\{Dove, Hawk\}$ and $\{Hawk, Dove\}$. The last case combines the pure strategy equilibria of the first and second cases. That is, for $c = 1$, the equilibria in pure strategies are $\{Hawk, Hawk\}$, $\{Dove, Hawk\}$, and $\{Hawk, Dove\}$.

To calculate the mixed equilibria, suppose that Animal *A* mixes its actions by choosing *Dove* with probability p . In equilibrium, Animal *B* should be indifferent between choosing *Dove* and *Hawk*, therefore,

$$p \frac{1}{2} = p + (1 - p) \frac{1}{2} (1 - c).$$

Solving for p gives

$$p = \frac{c - 1}{c}.$$

Thus, if $c < 1$, there is no mixed strategy equilibrium. If $c = 1$, then $p = 0$, so, we essentially get one of the pure strategy equilibria of this case (namely $\{Hawk, Hawk\}$). Lastly, if $c > 1$, we get an additional mixed equilibrium.

- c. If $c < 1$, the cost of fighting is small, and the animals attempt to get the value for their own. This leads to a prisoners' dilemma type of game with a Pareto inefficient outcome. If $c > 1$, the cost of fighting is great enough to deter fighting. The animals then resort to playing a coordination game (Similar to Bach or Stravinsky).
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3. Simultaneous Competition

3.1. Group A

1. Consider a Cournot duopoly with inverse demand $p(q) = p_0 + p_1 q$ and costs $c_i(q) = c_{1,i} q$, for $i = \{1, 2\}$, where $p_0, c_1 > 0$ and $p_1 < 0$. What happens if costs are not symmetric? Which firm produces more? Which firm makes more profit?

Hints

The best responses become

$$q_i = \frac{c_{1,i} - p_0 - p_1 q_j}{2p_1}.$$

Equilibrium ceases to be symmetric. The equilibrium quantities are given by

$$q_i = \frac{2c_{1,i} - c_{1,j} - p_0}{3p_1}.$$

The symmetric equilibrium quantities are obtained as a special case from the last formula by setting $c_{1,i} = c_{1,j}$.

Firm i produces more than j if and only

$$\frac{2c_{1,i} - c_{1,j} - p_0}{3p_1} \geq \frac{2c_{1,j} - c_{1,i} - p_0}{3p_1},$$

which, because $p_1 < 0$, is equivalent to $c_{1,i} \leq c_{1,j}$. Thus, the firm with the lower cost produces more.

The total market quantity is

$$q = \frac{2c_{1,i} - c_{1,j} - p_0}{3p_1} + \frac{2c_{1,j} - c_{1,i} - p_0}{3p_1} = \frac{c_{1,i} + c_{1,j} - 2p_0}{3p_1},$$

and the market price

$$p(q) = \frac{c_{1,i} + c_{1,j} + p_0}{3}.$$

We can then calculate the profit of firm i as

$$\begin{aligned} \pi_i &= \left(\frac{c_{1,i} + c_{1,j} + p_0}{3} - c_{1,i} \right) \frac{2c_{1,i} - c_{1,j} - p_0}{3p_1} \\ &= -\frac{(2c_{1,i} - c_{1,j} - p_0)^2}{9p_1}. \end{aligned}$$

The firm that produces more makes the greatest profit. The easiest way to get this result is to rewrite profits as $\pi_i = -q_i^2 p_1$. Since $p_1 < 0$, we have $\pi_i \geq \pi_j$ if and only if $q_i \geq q_j$.

3.2. Group B

1. Consider a differentiated product duopoly with firms competing in prices. The market consists of a homogeneous unit mass of consumers. Each consumer $h \in [0, 1]$ has unit demand for commodity $i = 1, 2$. The (gross) valuation of each consumer for commodity i is v_i with $v_1 > v_2$. If the net (gross minus price) valuations of the two commodities are equal, commodity 1 gets all the demand. The marginal production costs of both firms are zero.
 - a. Suppose that each consumer can only buy one of the two products. Find the Nash equilibria.
 - b. Suppose that consumers can also buy both products in a bundle. The bundle has valuation v_b with $v_1 + v_2 > v_b > v_1$. Find the Nash equilibrium.
 - c. Calculate the consumer surpluses of the two cases and compare them.

Hints

- a. The Nash equilibrium is characterized by $p_1^* = v_1 - v_2 > 0$ and $p_2^* = 0$. We work our way to this result by showing that there is no profitable unilateral deviation. In the candidate equilibrium, the profits are $\pi_1^* = v_1 - v_2$ and $\pi_2^* = 0$.

Suppose that firm 2 deviates to $p_2 > 0$. We then have $p_1 < p_2 + v_1 - v_2$. This implies that $v_2 - p_2 < v_1 - p_1$, so the consumers prefer commodity 1. As a result, $\pi_2 = 0 = \pi_2^*$. Thus, firm 2 has no profitable deviation.

Consider the deviations of firm 1. There are two potential deviations. Firstly, suppose that firm deviates to $p_1 > v_1 - v_2 = p_2 + v_1 - v_2$. Then, commodity 2 is preferred, and firm 1 makes zero profit, which is less than π_1^* . Secondly, consider a deviation $p_1 < v_1 - v_2 = p_2 + v_1 - v_2$. Then, commodity 1 is preferred, and firm 1 earns $\pi_1 = p_1 < v_1 - v_2 = \pi_1^*$.

- b. In equilibrium, it holds

$$v_1 - p_1^{**} = v_2 - p_2^{**} = v_b - p_1^{**} - p_2^{**}$$

because at least one of the two firms has a profitable deviation otherwise. For example, suppose that $v_1 - p_1 < v_2 - p_2^{**}$. Then, only commodity 2 is consumed, and the profit of firm 1 is zero. Firm 1 can deviate to p_1^{**} and earn profit $p_1^{**} = v_b - v_2 > 0$. Similarly, other cases can be excluded.

- c. The consumer surplus of the market in part one is

$$C^* = v_1 - p_1^* = v_1 - v_1 + v_2 = v_2.$$

The consumer surplus of the market in part two is

$$C^{**} = v_1 - p_1^{**} = \underbrace{v_1 - v_b}_{<0} + v_2 > 0.$$

Therefore, $C^* > C^{**}$ and consumers are better off in the first case.

4. Dynamic Games

4.1. Group A

1. Consider a two-player game in which, at date 1, player A selects a number $x > 0$, and player B observes it. At date 2, simultaneously and independently, player A selects a number y_A and player B selects a number y_B . The payoff functions of the two players are $u_A = y_A y_B + x y_A - y_A^2 - \frac{x^3}{3}$ and $u_B = -(y_A - y_B)^2$. Find all the subgame perfect Nash equilibria.

Hints

We find the subgame perfect equilibrium using backward induction. At the last date, player **B** solves

$$\max_{y_B} \{-(y_A - y_B)^2\}$$

and gets the best response $y_B(y_A) = y_A$. At the same date, player **A** solves

$$\max_{y_A} \left\{ y_A y_B + x y_A - y_A^2 - \frac{x^3}{3} \right\}$$

to get the best response $y_A(y_B; x) = x$. Solving the system of best responses, we find that $y_A^* = y_B^* = x$ in equilibrium.

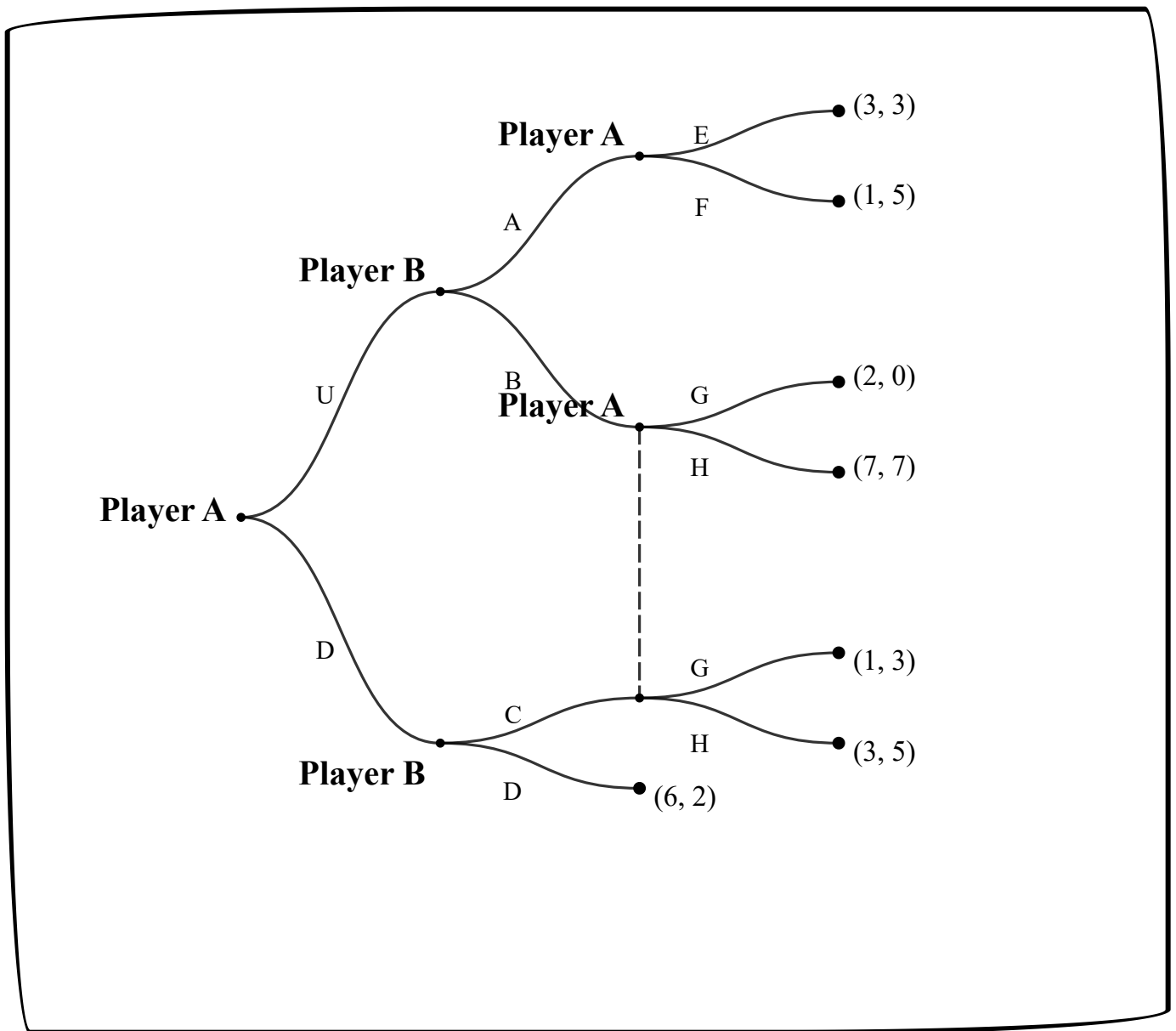
At the initial date, player **A** anticipates the behavior of date **2** and maximizes

$$\max_x \left\{ x^2 - \frac{x^3}{3} \right\}.$$

This gives the first order condition $2x - x^2 = 0$, from which we get the solution $x = 2$.

Consequently, we have a unique Nash equilibrium given by $\{(x = 2, y_A = 2), y_B = 2\}$.

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2. Find the subgame perfect equilibria of the following game. How many subgames does the game have?



Hints

There are two subgames. The subgame perfect equilibrium is $\{(U, E, H), (B, C)\}$.

4.2. Group B

1. Consider the alternating offer bargaining game with two players **A** and **B** negotiating how to split surplus with a unitary value. Player **A** moves first and makes an offer $x \in [0, 1]$. Player **B** moves second and decides whether to accept or reject the offer. If the offer is accepted, player **A** gets x , and player **B** gets $1 - x$. If the offer is rejected, player **B** makes a counteroffer $y \in [0, 1]$. If the counteroffer is accepted, player **A** gets $1 - y$, and player **B** gets y . If the counteroffer is rejected, the game continues with player **A** making an offer. Players discount every offer-round with factors $\delta_A, \delta_B \in [0, 1]$.
 - a. Calculate the recursive subgame perfect equilibrium.
 - b. Show that if the discount factor of player **A** increases, then she can get a greater part of the surplus.
 - c. Show that if the discount factor of player **B** increases, then player **A** gets a smaller part of the surplus.

Hints

- a. Suppose that we are at time t and player B makes an offer. Player A accepts the offer if and only if $1 - y \geq 0$. Thus, player B , aiming at maximizing her share, solves

$$\max_{1-y \geq 0} y$$

and offers $y = 1$. At time $t - 1$, player B knows that if the game proceeds to time t , player A will accept the offer $y = 1$. Therefore, if she waits for the next time point, player B can gain $\delta_B \cdot 1$ in present value terms. Hence, she accepts any offer of player A securing her at least this gain, namely $1 - x \geq \delta_B$. Player A anticipates this behavior, solves

$$\max_{1-x \geq \delta_B} x,$$

and offers $x = 1 - \delta_B$.

At time $t - 2$, player A knows that if she delays, she can gain $1 - \delta_B$, which corresponds to gaining $\delta_A(1 - \delta_B)$ discounted in time $t - 2$ terms. She, therefore, accepts an offer if $1 - y \geq \delta_A - \delta_A\delta_B$. Thus, player B solves

$$\max_{1-y \geq \delta_A - \delta_A\delta_B} y$$

and offers $y = 1 - \delta_A + \delta_A\delta_B$. In a similar manner, one argues that at time $t - 3$ player A solves

$$\max_{1-x \geq \delta_B - \delta_A\delta_B + \delta_A\delta_B^2} x,$$

and offers $x = 1 - \delta_B + \delta_A\delta_B - \delta_A\delta_B^2$.

Repeating the aforementioned recursive arguments, we find that player A 's offer at date $t - (2n - 1)$ (observe that, because we started with player B , player A moves at odd offer-rounds) is

$$\begin{aligned} x &= 1 - \delta_B + \delta_A\delta_B - \delta_A\delta_B^2 + \cdots + \delta_A^{n-1}\delta_B^{n-1} - \delta_A^{n-1}\delta_B^n \\ &= 1 + \cdots + \delta_A^{n-1}\delta_B^{n-1} - \delta_B(1 + \cdots + \delta_A^{n-1}\delta_B^{n-1}) \\ &= (1 - \delta_B) \sum_{j=0}^{n-1} \delta_A^j \delta_B^j. \end{aligned}$$

Letting n go to infinity, the share of player A converges to

$$x = \frac{1 - \delta_B}{1 - \delta_A\delta_B}.$$

- b. We can find the effect of δ_A on player A 's share by calculating

$$\frac{\partial x}{\partial \delta_A} = \delta_B \frac{1 - \delta_B}{(1 - \delta_A\delta_B)^2},$$

which is positive.

- c. From

$$\frac{\partial x}{\partial \delta_B} = -\frac{1 - \delta_A}{(1 - \delta_A\delta_B)^2} < 0,$$

we see that player A 's share is decreasing in δ_B .

5. Dynamic Competition

5.1. Group A

1. Consider an endogenous timing game in which firms simultaneously choose whether to play **Early** or to play **Late**. If both firms make the same choice, then they compete as in the Bertrand competition. If the firms make different choices, then they compete as in the Stackelberg model (sequential competition in quantities), where the leader is the firm that chose **Early** and the follower is the firm that chose **Late**. Suppose that firms have cost functions to $c(q) = q$, and market demand is given by $p(q) = 5 - q$.

- Calculate the profits of the firms under the different means of competition.
- Draw the extensive form of the game.
- Calculate all the Nash Equilibria.

Hints

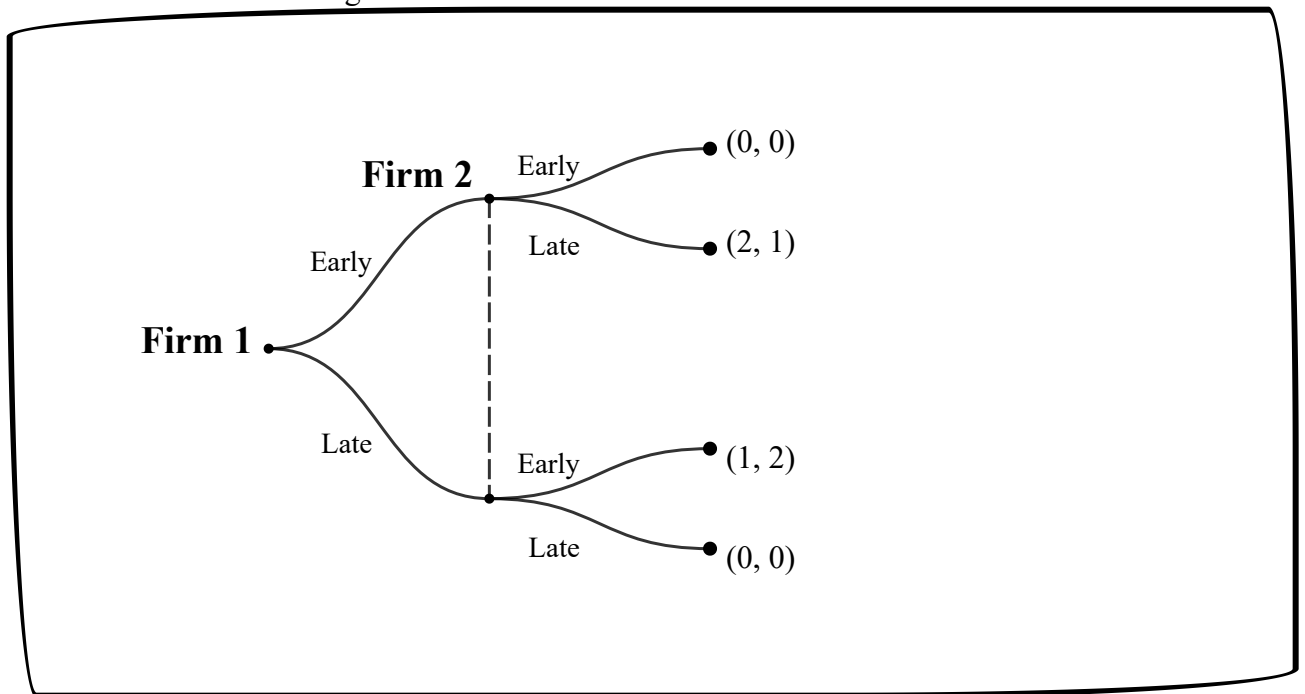
- If the firms compete simultaneously using prices, they set $p_1 = p_2 = 1$ and make zero profits. If they compete sequentially using quantities, the follower makes profit

$$\pi_f = \frac{(5 - 1)^2}{16} = 1,$$

and the leader makes profit

$$\pi_l = \frac{(5 - 1)^2}{8} = 2.$$

- The extensive form of the game is



- There are two pure strategy Nash equilibria, namely

$$\{\{Early, Late\}, \{Late, Early\}\},$$

and a mixed strategy equilibrium in which each firm chooses **Early** with probability $p = \frac{2}{3}$.

2. Suppose that there are two firms in a market with an affine inverse demand function $p(q) = p_0 + p_1 q$, where $p_0 > 0$ and $p_1 < 0$. The firms have constant marginal costs equal to c and no fixed production cost. Show that the leader in a leader-follower duopoly with quantity competition has a first move advantage. Specifically, show that the leader's profit is always greater than the profit it would attain in a market with simultaneous quantity competition.

Hints

In the leader-follower problem, the leader makes profit

$$\pi_l = -\frac{(c_1 - p_0)^2}{8p_1}.$$

In the simultaneous move game, each firm makes profit

$$\pi_s = -\frac{(c_1 - p_0)^2}{9p_1}.$$

One can calculate the ratio of the profits

$$\frac{\pi_l}{\pi_s} = \frac{(c_1 - p_0)^2}{8p_1} \frac{9p_1}{(c_1 - p_0)^2} = \frac{9}{8} > 1,$$

which implies that $\pi_l > \pi_s$.

3. Consider a market with inverse market demand function $p(q) = 100 - 2q$. The total cost function for any firm in the market is given by $c(q) = 4q$. Suppose that the market consists of two firms that compete by choosing their supplied quantities. Firm 1 is the market leader, while firm 2 is the follower. Calculate the firms' optimally supplied quantities, total market output, and market price.

Hints

The follower takes prices as given and solves

$$\max_{q_2} \{p(q_1 + q_2)q_2 - c(q_2)\},$$

which determines the best response of the follower as

$$b_2(q_1) = \frac{48 - q_1}{2}.$$

Then, the leader solves

$$\max_{q_1} \left\{ p \left(q_1 + \frac{48 - q_1}{2} \right) q_1 - 4q_1 \right\} = \max_{q_1} \{(48 - q_1)q_1\},$$

which determines the optimal supplied quantity of the leader as $q_1 = 24$. Substituting into the best response of the follower results in $q_2 = 12$. The market quantity is $q_s = q_1 + q_2 = 36$, and the market price is

$$p(q_s) = 100 - 72 = 28.$$

5.2. Group B

1. Consider the quantity competition with costly entry game. Suppose that firms face market demand $p(q) = p_0 + p_1 q$, have symmetric production costs $c(q) = c_1 q$, and entry costs $\bar{c}$, where $p_0, c_1 > 0$ and $p_1 < 0$. The social welfare as a function of the number of firms is given by

$$W(n) = \int_0^n p(sq(s)) (q(s) + sq'(s)) ds - nc(q(n)) - n\bar{c}.$$

Calculate the Pareto optimal number of entrants in the market.

Hints

- a. By Fermat's theorem, the welfare is maximized when $W'(n) = 0$. Using the fundamental theorem of calculus, we get

$$p(nq(n)) (q(n) + nq'(n)) - c(q(n)) - nc'(q(n))q'(n) = \bar{c}.$$

The optimal quantity produced by each firm when n firms enter the market is

$$q(n) = -\frac{p_0 - c_1}{(n+1)p_1},$$

hence, one calculates

$$\frac{q'(n)}{q(n)} = \frac{1}{n+1}.$$

We will use the last expression to simplify the first order condition. Rearranging the necessary condition and using the linearity of costs gives

$$\begin{aligned} \bar{c} &= p(nq(n)) (q(n) + nq'(n)) - c(q(n)) - nc'(q(n))q'(n) \\ &= p(nq(n))q(n) - c(q(n)) + n (p(nq(n)) - c'(q(n)))q'(n) \\ &= \pi(n) + n (p(nq(n))q(n) - c(q(n))) \frac{q'(n)}{q(n)} \\ &= \pi(n) \left(1 + n \frac{q'(n)}{q(n)} \right) \\ &= \pi(n) \left(1 - n \frac{1}{n+1} \right) \\ &= \pi(n) \left(\frac{1}{n+1} \right). \end{aligned}$$

The profit of each firm when n firms enter the market is

$$\pi(n) = -\frac{(p_0 - c_1)^2}{(n+1)^2 p_1},$$

so, we get

$$\bar{c} = -\frac{(p_0 - c_1)^2}{(n+1)^3 p_1}.$$

Therefore, the socially optimal number of firms is given by

$$n = \sqrt[3]{-\frac{(p_0 - c_1)^2}{p_1 \bar{c}}} - 1.$$

6. Collusion and Market Power

6.1. Group A

1. Consider two firms competing in quantities for an infinite number of dates. The discount factor is $\delta \in (0, 1)$. The market's inverse demand is $p(q) = 220 - q$, and both firms have a marginal cost equal to 10
 - a. Find the equilibrium under non cooperation (Cournot competition) at each date. Calculate the quantity that a monopolist would produce at each date. Calculate the profits in both cases and compare them.
 - b. Write down the trigger strategies of the two firms that can lead to tacit collusion.
 - c. What is the most profitable deviation from the trigger strategy at the initial date? What is the profit of the deviation at this date? How does the profit compare with the coordination profit of the trigger strategy?
 - d. For which values of δ is collusion sustainable?

Hints

- a. In the stage game, each firm maximizes

$$\max_{q_i} (220 - q_i - q_j - 10)q_i,$$

which implies that $q_i = q_j = q_c = 70$. When firms do not cooperate, they choose to produce these quantities at all dates. In this case, the market price is

$$p(2q_c) = 220 - 140 = 80,$$

and each firm makes profit

$$\pi_c = (80 - 10)70 = 4900.$$

The monopolist would instead maximize

$$\max_q (220 - q - 10)q,$$

which gives $q_m = 105$. The market price is then

$$p(q_m) = 220 - 105 = 115,$$

and the monopolist makes profit

$$\pi_m = (115 - 10)105 = 11025.$$

- b. The trigger strategy that can lead to collusion is for each firm i to produce according to the rule

$$q_{i,t} = \begin{cases} \frac{q_m}{2} & q_{j,s} = \frac{q_m}{2} \text{ for all } s < t \\ q_c & \text{otherwise} \end{cases}.$$

- c. Given that firm j plays the trigger strategy, firm i maximizes

$$\max_{q_i} \left(220 - q_i - \frac{105}{2} - 10 \right) q_i,$$

which implies that the best firm i can do, if it deviates from the trigger strategy, is to produce $q_d = \frac{315}{4}$. The profit of firm i is then

$$\pi_d = \frac{99225}{16},$$

which is greater than $\pi_m/2$. Thus, the firm can make greater flow profit at the initial date if it deviates from coordination.

- d. This practice, however, does not make the firm overall better off. Only short-sighted firms prefer this scenario. Firms that value future profits prefer coordination. Specifically, coordination can be sustained if and only if

$$\frac{1}{1-\delta} \frac{\pi_m}{2} \geq \pi_d + \frac{\delta}{1-\delta} \pi_c$$

or equivalently

$$\delta \geq \frac{\frac{\pi_m}{2} - \pi_d}{\pi_c - \pi_d} = \frac{\frac{11025}{2} - \frac{99225}{16}}{4900 - \frac{99225}{16}} = \frac{11025}{20825} = \frac{9}{17}.$$

2. Consider two firms competing in prices for a potentially infinite number of dates. The two firms have marginal costs equal to c . If the two firms set the same price, they split the market so that firm 1 gets a share μ and firm 2 gets a share $1 - \mu$. The firms do not discount future profits, but there is a probability $p > 0$ that the market closes down after each date. The firms use trigger strategies to coordinate and split the monopolistic profit.

- Find the equilibrium if $p = 1$.
- Suppose that $\mu = \frac{1}{2}$. Find the values of $p < 1$ for which collusion is sustainable. How do you interpret the result?
- Find the values of $p < 1$ for which collusion is sustainable for any value of $\mu \in (0, 1)$. What is the equilibrium relationship between p and μ ? How do you interpret the result?

Hints

- If $p = 1$, the firms compete only for one date, and the game reduces to Bertrand competition. The firms set $p_1 = p_2 = c$.
- The most profitable deviation of each firm at the initial date is to slightly undercut the monopoly price by ϵ . By doing so, the firm can take all the market demand and achieve almost the monopolistic profit. For simplicity, assume that the deviating firm earns exactly the monopolistic profit. Then, the deviation is not profitable if and only if

$$\frac{1}{p} \frac{\pi_m}{2} \geq \pi_m \iff p \leq \frac{1}{2}.$$

Collusion is sustainable if the probability that the market closes down is small and the firms have the potential to benefit from future profits.

- With arguments analogous to the $\mu = \frac{1}{2}$ case, the condition for firm 1 is

$$\frac{1}{p} \mu \pi_m \geq \pi_m \iff p \leq \mu,$$

and for firm 2 is

$$\frac{1}{p}(1 - \mu)\pi_m \geq \pi_m \iff p \leq 1 - \mu.$$

Therefore, collusion is sustainable if and only if

$$p \leq \min\{\mu, 1 - \mu\}.$$

If $\mu < \frac{1}{2}$, the equilibrium condition becomes less stringent when μ increases. In contrast, the equilibrium condition becomes more stringent when μ increases for $\mu > \frac{1}{2}$. The most lenient equilibrium condition is obtained for $\mu = \frac{1}{2}$. The further is μ from $\frac{1}{2}$, the less are the flow profits that one of the two firms makes and, as a result, coordination based on future benefits becomes less profitable.

6.2. Group B

1. Consider two firms competing in quantities for an infinite number of dates. The discount factor is $\delta \in (0, 1)$. The market's inverse demand is $p(q) = p_0 + p_1 q$ and both firms have marginal cost equal to c , where $p_0 > c > 0$ and $p_1 < 0$. Suppose that firms use the following tit-for-tat strategy. If the other firm has produced half the monopolistic quantity last date, then produce half the monopolistic quantity at this date. Otherwise, produce the Cournot quantity.

- a. Find the equilibrium under non cooperation (Cournot competition) at each date. Calculate the profits of the firms.
- b. Find the total discounted profits if each firm produces half the monopolistic quantity at each date.
- c. Find the most profitable deviation of firm i at the initial date. Calculate the profit of the deviating firm at the initial date.
- d. Suppose that at the second date, the deviating firm reverts to producing half the monopolistic quantity. Calculate the profit it makes at this date.
- e. For which values of δ is collusion sustainable?

Hints

- a. As in the grim trigger strategy case, if the firms do not coordinate, they produce

$$q_c = \frac{c - p_0}{3p_1},$$

make flow profits

$$\pi_c = -\frac{(p_0 - c)^2}{9p_1},$$

and total discounted profits

$$V_c = -\frac{1}{1 - \delta} \frac{(p_0 - c)^2}{9p_1}.$$

- b. In this case, each firm produces

$$\frac{q_m}{2} = \frac{c - p_0}{4p_1},$$

makes flow profit

$$\frac{\pi_m}{2} = -\frac{(p_0 - c)^2}{8p_1},$$

and total discounted profit

$$V_m = -\frac{1}{1-\delta} \frac{(p_0 - c)^2}{8p_1}.$$

c. The most profitable deviation of firm i is calculated by solving

$$\max_{q_i} \left(p_0 + p_1 \left(q_i + \frac{q_m}{2} \right) - c \right) q_i,$$

which leads to

$$q_1 = \frac{3}{8} \frac{c - p_0}{p_1},$$

and flow profit

$$\pi_1 = -\frac{9}{64} \frac{(c - p_0)^2}{p_1}.$$

d. Firm i is producing half the monopolistic quantity, and firm j is producing the Cournot quantity, punishing the initial date deviation. Thus, firm i makes profit

$$\begin{aligned} \pi_2 &= \left(p_0 + p_1 \left(\frac{q_m}{2} + q_c \right) - c \right) \frac{q_m}{2} \\ &= -\frac{5}{48} \frac{(c - p_0)^2}{p_1}. \end{aligned}$$

e. There are two cases we should consider. The deviating firm may revert back to producing half the monopolistic quantity after the deviation, or it can resort to producing the Cournot quantity. In the first case, the payoff is

$$V_r = \pi_1 + \delta\pi_2 + \delta^2 V_m,$$

while in the second case, it is

$$V_n = \pi_1 + \delta V_c.$$

The second case is excluded if and only if

$$\begin{aligned} V_r \geq V_n &\iff \pi_2 + \delta V_m \geq V_c \\ &\iff -\frac{5}{48} \frac{(c - p_0)^2}{p_1} - \frac{\delta}{1-\delta} \frac{1}{8} \frac{(c - p_0)^2}{p_1} \geq -\frac{1}{1-\delta} \frac{1}{9} \frac{(c - p_0)^2}{p_1} \\ &\iff \delta \geq \frac{1}{3}. \end{aligned}$$

Moreover, for collusion to be sustainable, we need

$$\begin{aligned} V_m \geq V_r &\iff V_m \geq \pi_1 + \delta\pi_2 + \delta^2 V_m \\ &\iff -\frac{1-\delta^2}{1-\delta} \frac{1}{8} \frac{(c - p_0)^2}{p_1} \geq -\frac{9}{64} \frac{(c - p_0)^2}{p_1} - \delta \frac{5}{48} \frac{(c - p_0)^2}{p_1} \\ &\iff (1+\delta) \frac{1}{8} \geq \frac{9}{64} + \delta \frac{5}{48} \\ &\iff \delta \geq \frac{3}{4}. \end{aligned}$$

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